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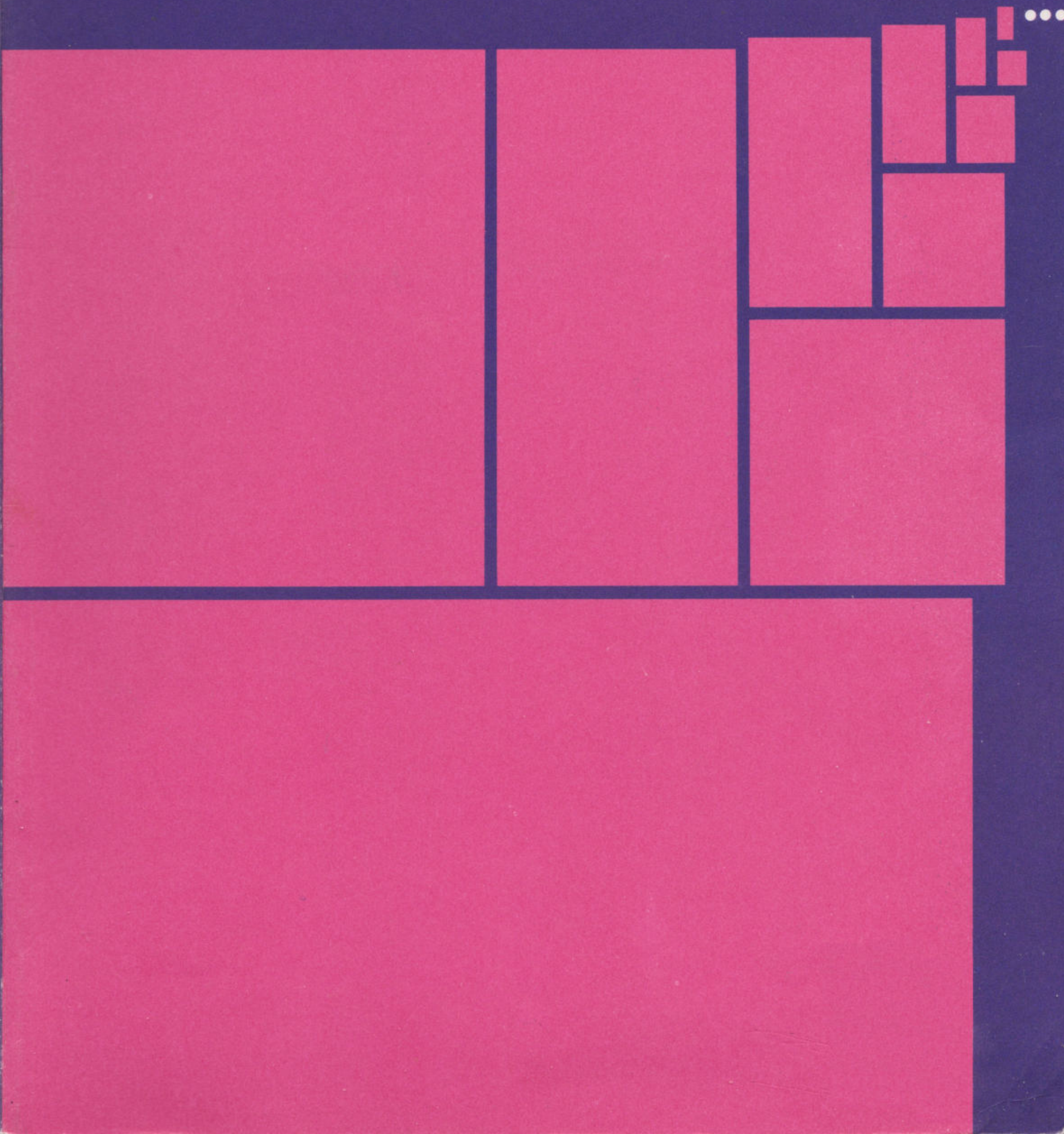
THE OPEN UNIVERSITY

Mathematics Foundation Course Unit 14

Sequences and Limits II



The Open University

Mathematics Foundation Course Unit 14

SEQUENCES AND LIMITS II

Prepared by the Mathematics Foundation Course Team

Correspondence Text 14

The Open University Press

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Contents

Objectives

After working through this unit you should be able to:

(i) write down the tangent approximation to the image value of a suitable function;

(ii) explain how the tangent approximation can be used to obtain a criterion for the convergence of the iterative method for solving an equation of the form $x = F(x)$;

(iii) explain and apply the Newton–Raphson process for the solution of an equation of the form $f(x) = 0$;

(iv) write down the Taylor and Maclaurin approximation formulas;

(v) calculate the first few terms (and the general term in simple cases) in the Taylor expansion of a given function about a given point in the domain of the function;

(vi) write down the statement of Taylor's Theorem, and use the theorem to estimate the error in a given Taylor expansion;

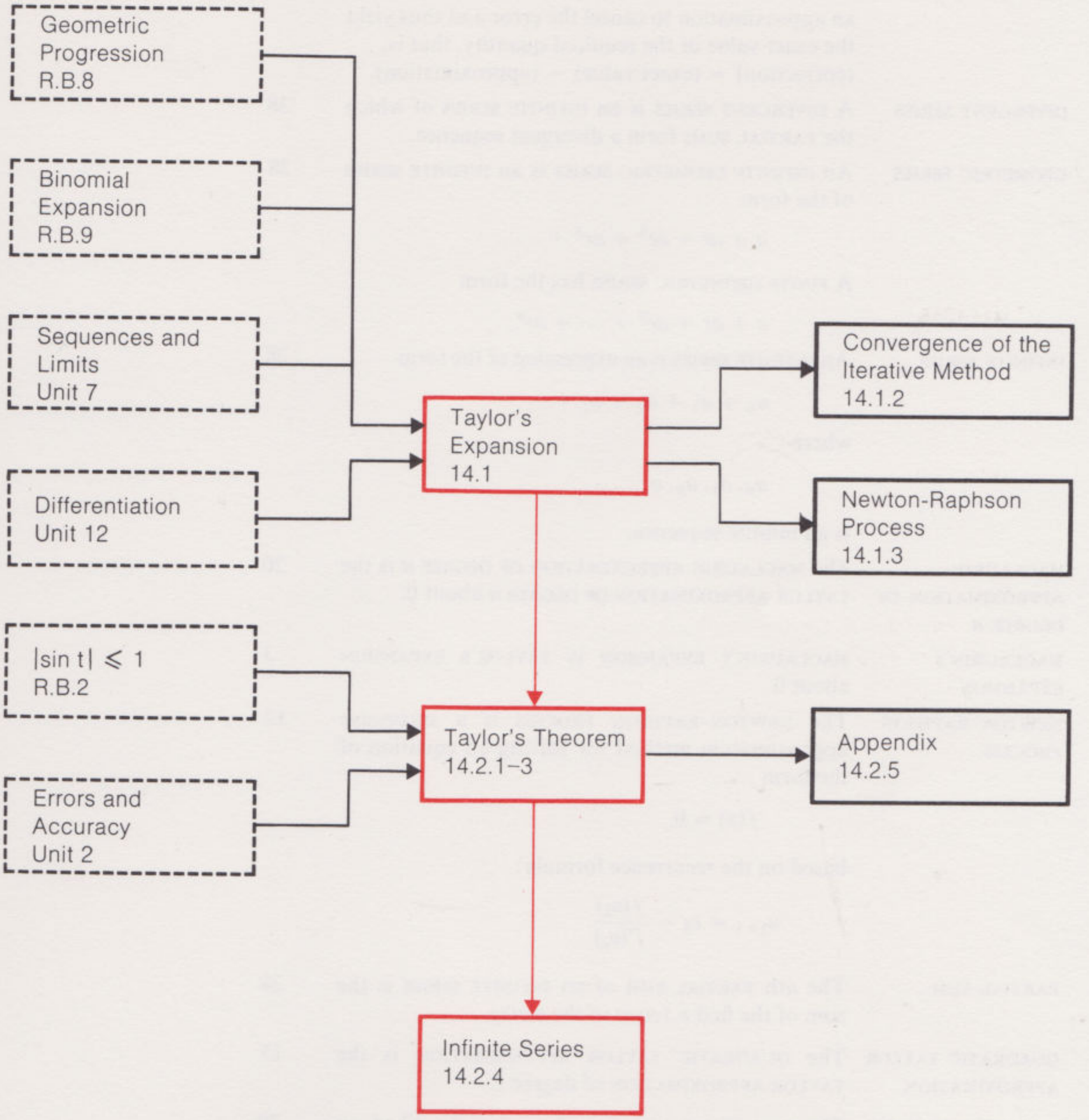
(vii) write down the definitions of the terms:
- infinite series
- partial sum of an infinite series
- convergent infinite series
- sum of a convergent infinite series;

(viii) decide whether a given simple infinite series is convergent or divergent.

Note

Before working through this correspondence text, make sure you have read the general introduction to the mathematics course in the Study Guide, as this explains the philosophy underlying the whole course. You should also be familiar with the section which explains how a text is constructed and the meanings attached to the stars and other symbols in the margin, as this will help you to find your way through the text.

Structural Diagram



Glossary

Terms which are defined in this glossary are printed in CAPITALS.

Term	Definition	Page
CONVERGENT SERIES	A CONVERGENT SERIES is an INFINITE SERIES of which the PARTIAL SUMS form a convergent sequence.	38
CORRECTION	A CORRECTION is the term which must be added to an approximation to cancel the error and thus yield the exact value of the required quantity, that is, (correction) = (exact value) − (approximation).	27
DIVERGENT SERIES	A DIVERGENT SERIES is an INFINITE SERIES of which the PARTIAL SUMS form a divergent sequence.	38
GEOMETRIC SERIES	An INFINITE GEOMETRIC SERIES is an INFINITE SERIES of the form $$a + ar + ar^2 + ar^3 + \cdots.$$ A FINITE GEOMETRIC SERIES has the form $$a + ar + ar^2 + \cdots + ar^n.$$	38
INFINITE SERIES	An INFINITE SERIES is an expression of the form $$a_0 + a_1 + a_2 + a_3 + \cdots$$ where $$a_0, a_1, a_2, a_3, \ldots$$ is an infinite sequence.	38
MACLAURIN APPROXIMATION OF DEGREE n	The MACLAURIN APPROXIMATION OF DEGREE n is the TAYLOR APPROXIMATION OF DEGREE n about 0.	20
MACLAURIN'S EXPANSION	MACLAURIN'S EXPANSION is TAYLOR'S EXPANSION about 0.	3
NEWTON–RAPHSON PROCESS	The NEWTON–RAPHSON PROCESS is a successive approximation method for solving an equation of the form $$f(x) = 0,$$ based on the recurrence formula: $$u_{k+1} = u_k - \frac{f(u_k)}{f'(u_k)}.$$	12
PARTIAL SUM	The nth PARTIAL SUM of an INFINITE SERIES is the sum of the first n terms of the series.	38
QUADRATIC TAYLOR APPROXIMATION	The QUADRATIC TAYLOR APPROXIMATION is the TAYLOR APPROXIMATION of degree 2.	15
SNOWFLAKE CURVE	The SNOWFLAKE CURVE is the curve described on page 39.	39
SUM	The SUM of a CONVERGENT SERIES is the limit of the sequence of PARTIAL SUMS of the INFINITE SERIES.	38
TANGENT APPROXIMATION	The TANGENT APPROXIMATION corresponds to approximating the pictorial graph of a function by a tangent to the graph at some suitable point.	6
TAYLOR APPROXIMATION OF DEGREE n	The TAYLOR APPROXIMATION OF DEGREE n about a is the approximation to a function f by the polynomial of degree n: $$x \longmapsto f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!}f^{(n)}(x-a)^n.$$	20

		Page
TAYLOR'S EXPANSION	TAYLOR'S EXPANSION is a method of obtaining successive polynomial approximations to certain images under a function, using the images under the function and its derived functions of a *single point* in the domain.	3
TAYLOR'S THEOREM	TAYLOR'S THEOREM is a theorem which gives a formula for the CORRECTION to the general TAYLOR APPROXIMATION.	28

Notation

Page

The symbols are presented in the order in which they appear in the text.

exp	The exponential function.	1
$\simeq$	"is approximately equal to".	2
$f'(a)$	The derivative of f at a.	6
$f''(a)$	The derivative of f' at a.	15
e	$e = \exp(1) = 2.71828\ldots$	15
Δ_h	The finite difference operator.	16
Δ_h^2	$\Delta_h \circ \Delta_h$.	16
D	The differentiation operator.	16
D^n	$\underbrace{D \circ D \circ \cdots D}_{n \text{ terms}}$	17
$f^{(n)}$	The nth derived function of f, $D^n f$.	19
$k!$	Factorial k; that is, $1 \times 2 \times 3 \times \cdots \times k$, where $k \in Z^+$.	20
$C_1(x)$	The correction to the tangent approximation for $f(x)$ about some given point.	27
$\wedge$	The symbol for conjunction ("and").	30
$\vee$	The symbol for alternation ("or").	30
$\Rightarrow$	The symbol for implication ("implies").	30
$\Leftrightarrow$	The symbol for equivalence ("implies and is implied by").	30
$C_n(x)$	The correction to the Taylor approximation of degree n for $f(x)$ about some given point.	31
S_k	The kth partial sum (the sum of the first k terms) of an infinite series.	38
$\int_a^b f$	The definite integral of f in $[a, b]$.	41
$[F]_a^b$	$F(b) - F(a)$.	43
ln	The logarithm function.	44

Bibliography

T. M. Apostol, *Calculus Vol. I* (Blaisdell, 1967).
Apostol discusses the fitting of the Taylor polynomial on pages 272 to 283. He derives a slightly different formula for the correction, $C_n(x)$ (in our notation); he calls this correction term "the remainder" and denotes it by $R(x)$. The definition of the sum of a convergent infinite series is given on page 384.

R. Courant, *Differential and Integral Calculus Vol. I* (Blackie, 1966).
Courant discusses the fitting of the Taylor polynomial on pages 315 to 329, covering roughly the same ground as Apostol, but in more detail and at a more leisurely pace. The sum of a convergent infinite series is discussed briefly on pages 366 to 368.

Like most books on calculus, both these books discuss various criteria for the convergence of infinite series, and they also give many examples of the use of Taylor's and Maclaurin's series for particular functions, together with rules for adding, multiplying, dividing, differentiating and integrating infinite series. The subject of infinite series and the representation of functions by convergent sequences of polynomial approximations is a very big one, of which only a small fraction can be dealt with in our Foundation Course.

14.0 INTRODUCTION

14.0

Introduction

In this unit we return to a problem which we considered in *Unit 4, Finite Differences*: how to evaluate the images of real functions which cannot be expressed in terms of the elementary operations of arithmetic. In *Unit 4* we based our discussion on the assumption that a table of images was available; it was then possible to approximate to the function (over a suitable interval of the domain) by a linear or polynomial function chosen to fit the function at the tabular points. In many cases, however, the tabulated values necessary for this method may not exist; and in any case, where do the numbers in the table come from? Even when the tables do exist, they may not provide the most suitable source of image values for the function. In particular, automatic computers cannot afford the space to store tables of functions; it is much more economical to store a sub-routine that calculates particular images from scratch as they are required (just as you might find it more convenient to work out the cost of $7\frac{1}{2}$ yards of curtain material at £1.71 a yard with pencil and paper rather than hunt around for a ready reckoner).

The type of approximation method we shall develop here is similar to the methods we used in *Unit 4, Finite Differences* in that it is based on polynomial approximation. However, instead of being fitted to the images of a function corresponding to equally spaced tabular points, the polynomials are fitted by a different method, which can be regarded as the limiting form of the finite difference method for very small tabular spacing. Instead of requiring a knowledge of the values of the images of the function at a set of equally spaced tabular points, the new method requires a knowledge of the value of the function at just one point, together with its derivative, and perhaps also some of the higher derivatives (derivatives of derivatives) at the same point. The virtue of the new method is that it may be much easier to calculate a number of higher derivatives at a single point in the domain of the function than to calculate images of the function at a number of different points. In fact, for this reason, the new method plays an important part in the calculation of the tabular values which are used in the method of approximation described in *Unit 4*.

We have already mentioned the use of this new method in computing images of the function; it has many other uses too. It is used, for example, in studying the effectiveness of numerical approximation methods, such as the iterative methods for solving equations of the form $x = f(x)$ which were described in *Unit 2, Errors and Accuracy*, and investigated further in *Unit 7, Sequences and Limits I*. In *Unit 2* we discussed a way of discovering in advance whether the recurrence formula $u_n = F(u_{n-1})$, defining the iteration sequence, would or would not give a convergent sequence of approximations to the solution of the equation, using the idea of a scale factor. But this method is not always simple in practice, and to simplify the discussion of the convergence of the iteration it is useful to have a simple approximation to $F(u)$ when u is close to the supposed limit of the iterative sequence: that is, to the solution of the equation $x = f(x)$. The method we shall describe in this unit provides such an approximation, and leads to a simple criterion for picking out the cases for which the iterative method converges.

In addition to these applications to computing, the method we shall describe is valuable as a method of *defining* functions. For example, the definition of the exponential function given in *Unit 7, Sequences and Limits I*:

$$\exp : x \longmapsto \lim_{k \text{ large}} \left(1 + \frac{x}{k}\right)^k \qquad (x \in R)$$

is by no means universal; many people prefer to define the exponential

function in terms of the limit of a sequence of polynomial functions of the type which we discuss in this correspondence text (though, of course, all the definitions define the same function). For the exponential function this may not seem particularly useful, since the definition already given is perfectly adequate; but mathematicians meet many functions which are more complicated than the exponential function, and for these the definition as the limit of a sequence of polynomials is often the most convenient. In fact this new method of defining functions was the starting point of developments which enormously expanded the scope of mathematics during the eighteenth century. Some of these developments will be mentioned in the radio component of this unit.

14.1 TAYLOR'S EXPANSION

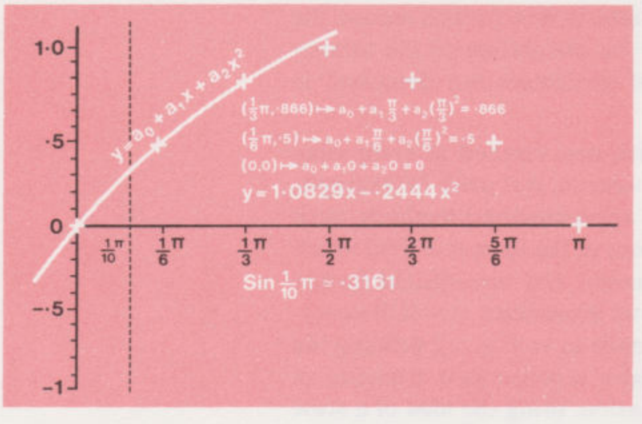
14.1

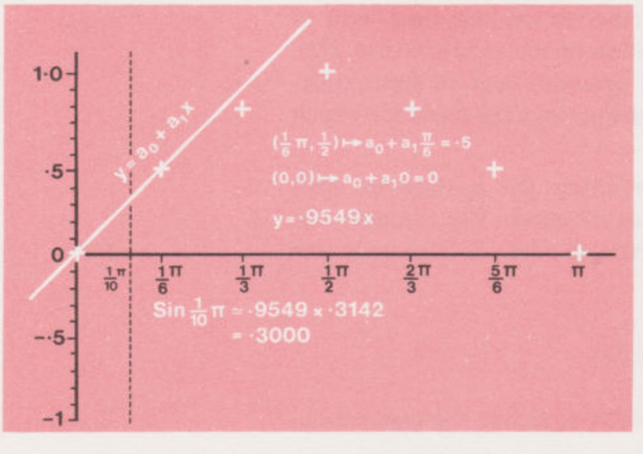
14.1.0 Introduction

14.1.0

Introduction

Taylor's expansion is a method of obtaining successive approximations to the images of suitable elements in the domains of certain functions. A typical case is the evaluation of $\sin(18°) = \sin\left(\frac{\pi}{10}\right)$, which is treated in the television component of this unit. The first method tried out there is to fit polynomials to known points on the graph of the sine function. Both linear and quadratic approximations are tried, but neither gives better than a 2% approximation to the true value, 0.3090 (to four decimal places).





The reason for this low accuracy is essentially the large tabular spacing used. We could try to improve the accuracy by using interpolation polynomials of higher degree, or by using a smaller tabular interval, say $\frac{\pi}{12}$ or even $\frac{\pi}{24}$ instead of $\frac{\pi}{6}$. But both methods are cumbersome, the first because one has to use polynomials of very high degree to get high accuracy in the sines, and the second because the trigonometric formulas from which we can calculate $\sin\frac{\pi}{12}$, $\sin\frac{\pi}{24}$ and so on are rather complicated.

A further disadvantage of this method is that it lacks generality: we happen to know some special images under the sine function, but for other functions, such as the exponential function or the logarithm function, there is no such ready-made source of image values from which to make a table. It would be very convenient to have a method that did not require at the outset a set of tabular values of the function.

The polynomial approximation method which we shall discuss in this text provides such a method; the polynomials are fitted using the images under the function and its various derived functions (first, second, third, etc.), of a *single point* in the domain, instead of images under the function alone of *several different points* in the domain. The method is called Taylor's expansion. The special case of Taylor's expansion where the chosen element in the domain is the number 0 is called Maclaurin's expansion; this special case is the one treated in the television component of this unit.

Brook Taylor
1685–1731
(University College Library)

Colin Maclaurin
1698–1746
(Mansell Collection)

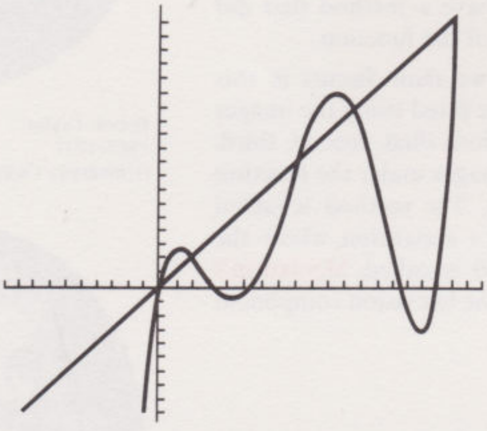
14.1.1 The Tangent Approximation

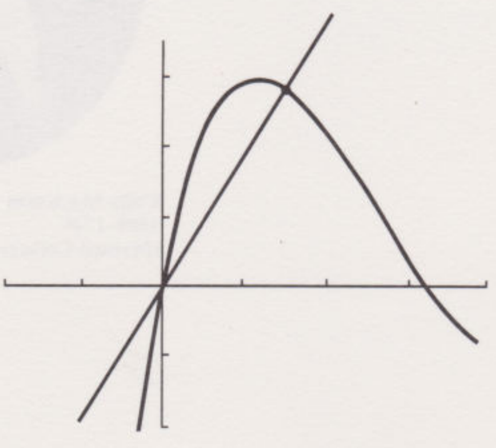
14.1.1

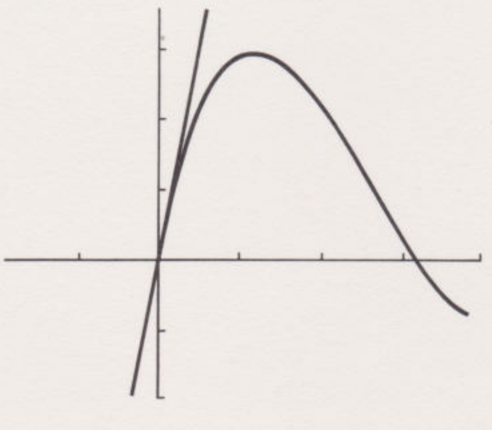
Main Text

* * *

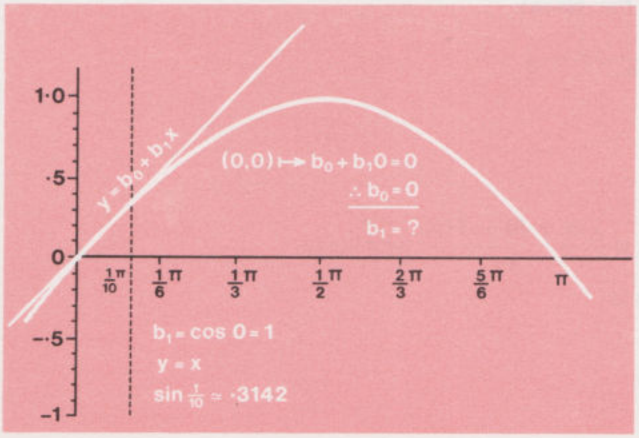
Just as in the theory of finite difference methods, where the simplest methods of interpolation and extrapolation are based on linear approximations, so the simplest form of Taylor's expansion is a linear approximation. One way of arriving at this approximation is to consider a straight line chosen to pass through two points very close together on the graph of the function. The diagrams from the television programme which are shown below illustrate how, in the limit as the tabular spacing h approaches zero, the straight line approaches a tangent, which, in this particular case, is the tangent at the point where the curve passes through the origin.







To calculate $\sin\left(\frac{\pi}{10}\right)$ by this method, we find the equation of the tangent to the curve at the nearest convenient point, which in this case is the origin.



If we assume that the equation of the tangent is $y = b_0 + b_1x$, then, since the tangent passes through the origin, we find $b_0 = 0$. Further, the derived function of sin is cos, and $\cos 0 = 1$, so that the slope of the tangent, b_1, is 1. Hence the equation of the tangent at the origin is

$$y = x$$

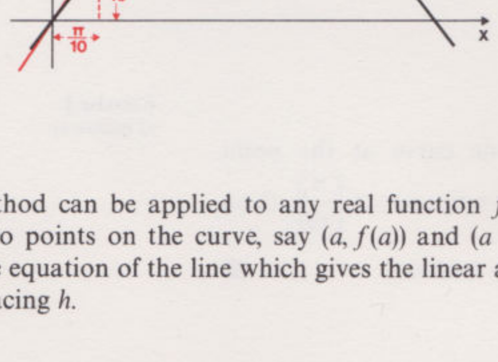
and our first (tangent) approximation to $\sin x$ is

$$\sin x \simeq x.$$

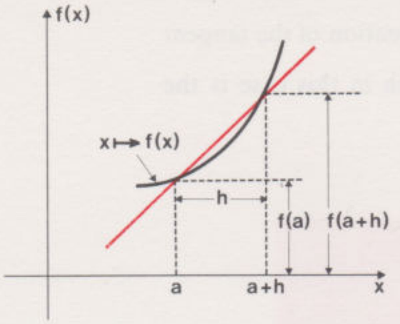
Thus

$$\sin\left(\frac{\pi}{10}\right) \simeq \frac{\pi}{10} = 0.3142.$$

(A method for calculating π is given in *Unit 13, Integration II.*)



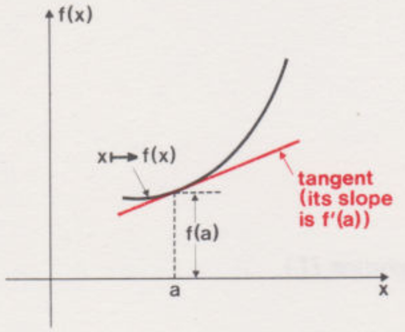
The same method can be applied to any real function f. We start by taking any two points on the curve, say $(a, f(a))$ and $(a + h, f(a + h))$, and obtain the equation of the line which gives the linear approximation for tabular spacing h.



The slope of the line joining the two points is $\dfrac{f(a+h)-f(a)}{h}$, and since the line passes through $(a, f(a))$ its equation is

$$y = f(a) + \frac{f(a+h)-f(a)}{h}(x-a).$$

We want the limiting form of this equation as the right-hand point approaches the left-hand one, that is, as h approaches zero. In this limit, the line still passes through the point $(a, f(a))$, but its slope is now the derivative at this point, namely $f'(a)$, (see section 12.1.4 of *Unit 12, Differentiation I*).



Accordingly, the equation of the tangent at $(a, f(a))$ is

$$y = f(a) + f'(a)(x-a).$$

Equation (1)

The tangent approximation is obtained by taking the right-hand side of Equation (1) as an approximation to $f(x)$; that is,

Definition 1

$$f(x) \simeq f(a) + f'(a)(x-a).$$

Exercise 1

Exercise 1
(2 minutes)

Find the equation of the tangent to the sine curve at the point $\left(\dfrac{\pi}{6}, \sin\dfrac{\pi}{6}\right)$, and use this as an approximation to estimate $\sin\left(\dfrac{\pi}{10}\right)$. (You may assume that $\sin\left(\dfrac{\pi}{6}\right) = 0.5000$ and $\cos\left(\dfrac{\pi}{6}\right) = 0.8660$.) ■

Exercise 2

Exercise 2
(4 minutes)

Would you expect the tangent approximation to give the greatest accuracy for large or small values of $|x - a|$? To test your answer, choose a value of a, and then calculate $\exp x$ from a tangent approximation for each of the values of x tabulated at the right; tabulate the error

x	$\exp x$
0.8	2.23
0.9	2.46
1.0	2.72
1.1	3.00
1.2	3.32

$$(\text{tangent approximation to } \exp x) - \exp x,$$

and note how it varies with x for a fixed value of a. ■

Exercise 3

Exercise 3
(4 minutes)

When a solid is heated it expands. The volume coefficient of thermal expansion of a solid may be defined as

$$\frac{\text{increase in volume due to a temperature increase of one degree}}{\text{original volume}}$$

and the linear coefficient of expansion may be defined as

$$\frac{\text{increase in any linear dimension due to a temperature increase of one degree}}{\text{original linear dimension}}.$$

For copper, the volume coefficient of expansion is about 50×10^{-6} per degree Centigrade, and the linear coefficient is 16×10^{-6} per degree Centigrade, which is about one third of the volume coefficient. Is this simple relation between the coefficients merely a coincidence? ■

Solution 1

Solution 1

From Equation (1), the equation of the tangent is

$$y = \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)$$

since $\sin' = \cos$. The approximation to $\sin\left(\frac{\pi}{10}\right)$ is therefore

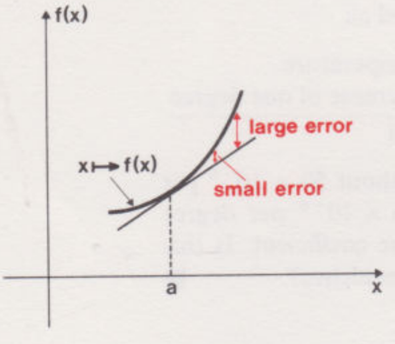
$$\begin{aligned}\sin\left(\frac{\pi}{10}\right) &\simeq \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\left(\frac{\pi}{10} - \frac{\pi}{6}\right)\\ &= 0.5000 + 0.8660(0.3142 - 0.5236)\\ &= 0.3187.\end{aligned}$$

(This approximation is about 3% larger than the correct value, 0.3090.)

■

Solution 2

Solution 2



From diagrams such as the above, we expect the approximation to improve as x approaches a. That is to say, the smaller $|x - a|$ is, the smaller will be the error in the approximation.
Choosing $a = 1$ (only because this value appears in the middle of the table), we obtain

$$\begin{aligned}\exp x &\simeq \exp 1 + \exp'(1)(x - a)\\ &= 2.72 + 2.72(x - 1),\end{aligned}$$

since

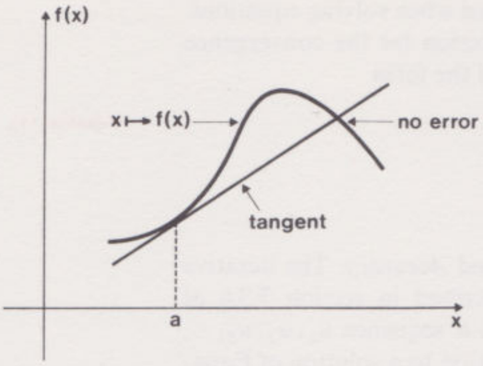
$$\exp'(1) = \exp(1) = 2.72.$$

The approximations and their errors are given in the following table:

x	$\exp x$	$x - a$	Tangent approxn.	Error
0.8	2.23	−0.2	2.18	−0.05
0.9	2.46	−0.1	2.45	−0.01
1.0	2.72	0	2.72	0
1.1	3.00	0.1	2.99	−0.01
1.2	3.32	0.2	3.26	−0.06

If you chose a different value for a, you should still have found the error larger, the larger the value of $|x - a|$. (In fact the error is roughly proportional to $(x - a)^2$.) Of course, in special cases the above argument may

not hold; for instance, if the curve "bends back" towards the tangent or intersects the tangent somewhere close to a. But, in general, "near to" a the above argument is sound, although intuitive.



■

Solution 3

Solution 3

No. Suppose we have a cube of any solid and we increase the temperature of the cube by 1 degree. If the length of the side of the cube is L, then the new length will be $L(1 + x)$, where x is the linear coefficient of expansion. Thus the new volume is $L^3(1 + x)^3$.

Thus the volume coefficient of expansion is

$$\frac{L^3(1 + x)^3 - L^3}{L^3} = (1 + x)^3 - 1.$$

Now if

$$f: x \longmapsto (1 + x)^3 - 1 \qquad (x \in R),$$

then

$$f': x \longmapsto 3(1 + x)^2 \qquad (x \in R).$$

So the tangent approximation to $(1 + x)^3 - 1$, using the tangent at $x = 0$, is

$$\begin{aligned}(1 + x)^3 - 1 &\simeq f(0) + f'(0)(x - 0)\\ &= 3x.\end{aligned}$$

Thus the volume coefficient of expansion is approximately three times the linear coefficient for *any* solid — and so the case of copper was not merely a coincidence. ■

14.1.2 Convergence of an Iterative Method

14.1.2

Discussion

Although the tangent approximation is not very accurate, it is simple to use, and can be very effective when the accuracy it offers is sufficient for the matter in hand. Before proceeding to discuss how we can improve its accuracy, we shall consider how it can be used when solving equations. In this section we shall use it to obtain a criterion for the convergence of the iterative method for solving equations of the form

$$x = F(x)$$

Equation (1)

such as the "omelette equation"

$$x = \sin x + \tfrac{2}{3}\pi$$

introduced in section 2.4.2 of *Unit 2, Errors and Accuracy*. The iterative method for solving such equations was described in section 7.3.4 of *Unit 7, Sequences and Limits I*: we construct a sequence $u_1, u_2, u_3, \ldots$ in which the first term is any crude approximation to a solution of Equation (1), and the later terms are calculated using the recurrence formula:

$$u_k = F(u_{k-1}) \qquad (k = 2, 3, 4, \ldots).$$

It was shown in *Unit 7* that, if this sequence converges to a limit a and F is continuous at a, then a is a solution of Equation (1). To avoid wasting time calculating the elements of non-convergent sequences, it is useful to have a simple criterion by which we can tell, without actually doing the calculation, which solutions of Equation (1) (if any) can be found by this method.

In *Unit 2, Errors and Accuracy* we discussed one way of finding such a criterion, but the tangent approximation gives a simpler method. To start with, we suppose that the sequence $u_1, u_2, \ldots$ does converge, and that its limit is a. Then, for large k, the numbers u_k are close to a, and so it is natural to consider using the tangent approximation to simplify the right-hand side of the recurrence formula

$$u_k = F(u_{k-1}) \qquad (k = 2, 3, \ldots).$$

Equation (2)

The tangent approximation for $F(u_{k-1})$ that is useful when u_{k-1} is close to a is

$$F(u_{k-1}) \simeq F(a) + F'(a)(u_{k-1} - a)$$

(see Equation 14.1.1.1).

Substituting this into Equation (2), we obtain

$$u_k \simeq F(a) + F'(a)(u_{k-1} - a).$$

Since a is a solution of the equation $x = F(x)$, we have $a = F(a)$, and so this last approximation is equivalent to

$$u_k - a \simeq F'(a)(u_{k-1} - a).$$

That is, when k is large, the deviation of the kth term, u_k, from the limit, a, is $u_k - a$, and differs from the preceding deviation, $u_{k-1} - a$, by a factor $F'(a)$, which is independent of k; that is,

$$k\text{th deviation} \simeq F'(a) \times ((k-1)\text{th deviation}).$$

It follows that, when k is large, the deviations will increase as k increases if $|F'(a)| > 1$. But if the sequence $u_1, u_2, \ldots$ converges to a limit a, then the deviations from a must eventually decrease as we take elements later and later in the sequence. Thus if the iterative sequence converges to a, then $|F'(a)| < 1$.

Notice the if in that last sentence. To complete the criterion it would be nice to be able to prove the converse statement: "if $a = F(a)$ and $|F'(a)| < 1$, then the iterative sequence converges to a". It is not quite

as simple as this, however; for example, there might be two different numbers a_1 and a_2, both being solutions of $x = F(x)$ and such that $|F'(a_1)| < 1$ and $|F'(a_2)| < 1$, but the sequence could not possibly converge to both of them, since the limit of a convergent sequence is *unique*. What we can say is that if $a = F(a)$ and $|F'(a)| < 1$, and u_1 is chosen close enough to a, then the sequence $u_1, u_2, \ldots$ will converge to a, for then the deviations $u_1 - a, u_2 - a, \ldots$ approximately form a geometric progression converging to zero. If, however, u_1 is chosen so far from a that the tangent approximation for $F(u_1)$ is very inaccurate, then we have no reason to expect the sequence to converge to a. It may ultimately converge to a anyway, but it may converge to some other solution of $x = F(x)$, or it may not converge at all. (See RB8)

Exercise 1

Exercise 1
(2 minutes)

The equation $x = x^2 + \frac{1}{2}x$ has two solutions. Without calculating iterative sequences, predict which of them can be computed using the iterative method based on the recurrence formula

$$u_k = u_{k-1}^2 + \tfrac{1}{2}u_{k-1}.$$

■

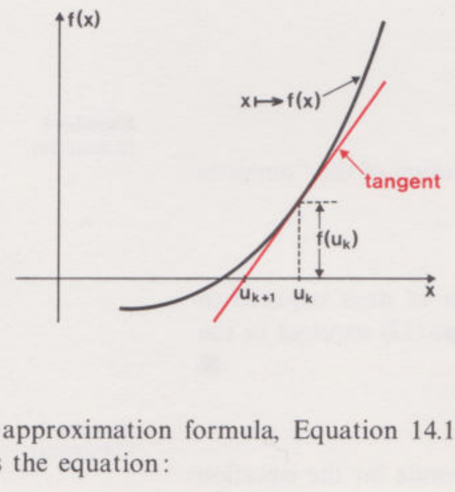
14.1.3 The Newton–Raphson Process

14.1.3

Discussion

As our second application of the tangent approximation in numerical methods, we shall use it to obtain a method for the numerical solution of equations which has very good convergence properties. The new method, known as the Newton–Raphson process, is again an iterative method, but here the tangent approximation is an integral part of the method, instead of being used almost as an afterthought to discuss the convergence.

Since we are not now interested in the iteration $u_n = F(u_{n-1})$, we shall not write the equation to be solved in the form $x = F(x)$, but in the more convenient form $f(x) = 0$. (The previous equation, $x = F(x)$, can be put into this form by taking $f(x) = x - F(x)$.) To construct the recurrence formula for the Newton–Raphson iteration, suppose that, after $k - 1$ steps of the iteration, the latest approximation to the solution of $f(x) = 0$ is u_k; we use the tangent approximation to f near u_k to estimate the value of x where $f(x) = 0$, and we take this estimate as our next approximation, u_{k+1}. The calculations are illustrated in the figure:



By the tangent approximation formula, Equation 14.1.1.1, the tangent at $(u_k, f(u_k))$ has the equation:

$$y = f(u_k) + f'(u_k)(x - u_k).$$

Equation (1)

(*continued on page 12*)

Solution 14.1.2.1 $x^2 + \frac{1}{2}x$ $2x + \frac{1}{2}$

Solution 14.1.2.1

Solving the quadratic equation directly, we obtain the solutions $x = 0$ and $x = \frac{1}{2}$. Convergence of the iterative sequence depends on the value of $|F'(a)| = |2a + \frac{1}{2}|$. If $a = 0$, $|F'(a)|$ is $\frac{1}{2}$, which is less than 1. So provided the initial guess is close enough to 0, the iterative method will work. If $a = \frac{1}{2}$, then $|F'(a)| > 1$, so that we cannot get the solution $x = \frac{1}{2}$ by the given recurrence formula, unless we choose a very lucky starting value. The following table gives a sample iteration.

	Sequence starting near 0	Sequence starting near $\frac{1}{2}$
u_1	0.1	0.6
u_2	0.06	0.66
u_3	0.0336	0.7656
u_4	0.0179	0.9689
u_5	0.00928	1.423
u_6	0.00473	2.738
u_7	0.00239	8.863
u_8	0.00120	82.979
	(converging to 0)	(diverging)

■

(*continued from page 11*)

While it may not be possible to solve the equation $f(x) = 0$ exactly (that is why we need numerical methods at all), there is no difficulty in solving the equation

$$(\text{linear approximation to } f(x)) = 0,$$

Equation (2)

because it is *linear*. Using the linear approximation on the right-hand side of Equation (1) in Equation (2), we obtain

$$f(u_k) + f'(u_k)(x - u_k) = 0$$

and the solution for x is

$$x = u_k - \frac{f(u_k)}{f'(u_k)}.$$

This is the value of x where the tangent approximation to $f(x)$ is 0, and so we use it as our next approximation to the value of x where $f(x)$ itself is 0. Accordingly, the recurrence formula for the Newton–Raphson method is

Definition 1

$$u_{k+1} = u_k - \frac{f(u_k)}{f'(u_k)}.$$

Exercise 1

Exercise 1
(2 minutes)

Apply the Newton–Raphson method to find a solution of the "omelette equation",

$$x = \sin x + \tfrac{2}{3}\pi,$$

lying between 2 and 3, and compare the number of steps required to achieve 3-figure accuracy with the number of steps (13) required in the method we used in *Unit 2*. ■

Exercise 2

Exercise 2
(2 minutes)

Write down the Newton–Raphson recurrence formula for the equation $x^2 - a = 0$. Does it look familiar? ■

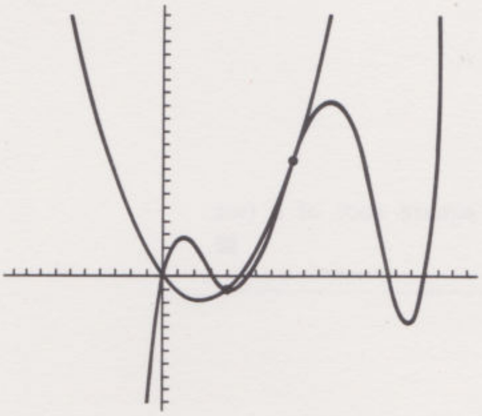
14.1.4 The Quadratic Taylor Approximation

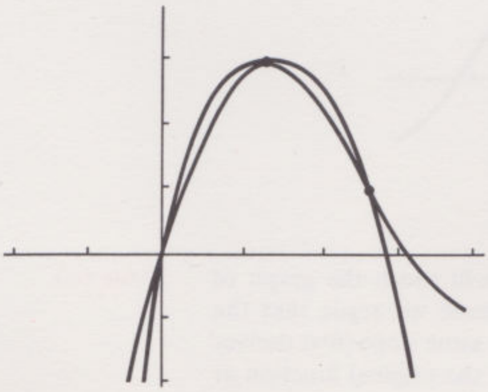
14.1.4

Discussion
* *

The iterative methods for solving $f(x) = 0$ based on the tangent approximation can usually yield any degree of accuracy provided we iterate for long enough. But they only work if we can calculate $f(x)$ for any x; they do not give us a method for calculating $f(x)$ itself. We have not yet found a way of calculating $\sin\left(\frac{\pi}{10}\right)$ to better than about 3% accuracy. The tangent approximation is very good near the point where the tangent touches the curve, but the accuracy falls off rapidly away from this point; $\frac{\pi}{10}$ is too far away from the points of contact in the tangent approximations which we have tried for $\sin\left(\frac{\pi}{10}\right)$. To improve the accuracy we need something better than the tangent approximation. As in the case of interpolation from tabulated values, one way to try to improve the approximation is to use quadratic, or even higher-degree polynomials, in place of the linear ones we have been using so far.

In the television component of this unit, we obtain the quadratic approximation by fitting a quadratic function of the form $x \longmapsto c_0 + c_1x + c_2x^2$, where c_0, c_1, c_2 are numbers, to the given function at equally spaced points in the domain, and then making the spacing h between these points extremely small. In the limit as h approaches 0, this quadratic approximation approaches the *quadratic Taylor approximation.*





(*continued on page 14*)

Solution 14.1.3.1

Solution 14.1.3.1

The equation is most conveniently put in the form $f(x) = 0$ by taking

$$f(x) = x - \sin x - \tfrac{2}{3}\pi.$$

Then $f'(x) = 1 - \cos x$, and the Newton–Raphson recurrence formula is:

$$u_k = u_{k-1} - \frac{u_{k-1} - \sin(u_{k-1}) - \frac{2}{3}\pi}{1 - \cos(u_{k-1})}.$$

The value of u_1 is not all that critical, but it is sensible to try to get it near to the solution. We have chosen $u_1 = 2$; you may well have chosen some other value, but you should get the same final result. Working to 3 decimal places, we obtain

$$u_1 = 2$$

$$u_2 = 2.709$$

$$u_3 = 2.607$$

$$u_4 = 2.605$$

$$u_5 = 2.605.$$

Only four iterations are required to give 2.605, compared with the 13 steps of our earlier method. ■

Solution 14.1.3.2

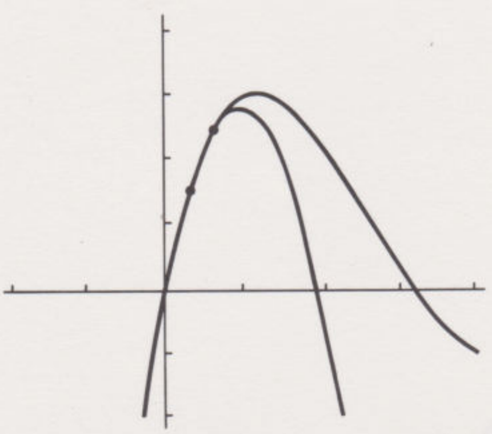
Solution 14.1.3.2

In this case $f(x) = x^2 - a$ and $f'(x) = 2x$. The Newton–Raphson recurrence formula is:

$$\begin{aligned} u_k &= u_{k-1} - \frac{u_{k-1}^2 - a}{2u_{k-1}} \\ &= u_{k-1} - \tfrac{1}{2}u_{k-1} + \frac{1}{2}\frac{a}{u_{k-1}} \\ &= \frac{1}{2}\left(u_{k-1} + \frac{a}{u_{k-1}}\right). \end{aligned}$$

This is *Newton's Formula* for calculating the square root of a (see section 7.1.2 of *Unit 7, Sequences and Limits I*). ■

(*continued from page 13*)



Main Text

The graph of this limiting quadratic function will touch the graph of the original function; in the television programme we argue that the graph of this quadratic function not only has the same slope (first derivative) but also has the same second derivative as the original function at

the point of contact. Denoting the quadratic function which we are using to approximate to f by q, and the value of x where the curves touch by a, the conditions to be satisfied are

$$\left.\begin{array}{ll} \text{equal images for } a\text{:} & q(a) = f(a) \\ \text{equal slopes at } a\text{:} & q'(a) = f'(a) \\ \text{equal second derivatives at } a\text{:} & q''(a) = f''(a) \end{array}\right\}$$

Equations (1)

These three conditions provide just sufficient information to determine the three coefficients c_0, c_1, c_2 in the expression $q(x) = c_0 + c_1x + c_2x^2$. The neatest way to use these conditions is to write the quadratic in the alternative form (analogous to the formula $f(a) + f'(a)(x - a)$ for the tangent approximation):

$$q(x) = b_0 + b_1(x - a) + b_2(x - a)^2.$$

Equation (2)

Differentiating, we get

$$q'(x) = b_1 + 2b_2(x - a)$$

$$q''(x) = 2b_2$$

so that the values of the quadratic function and its derivatives at a are

$$q(a) = b_0$$

$$q'(a) = b_1$$

$$q''(a) = 2b_2.$$

Comparing with Equations (1) we find that

$$b_0 = f(a)$$

$$b_1 = f'(a)$$

$$b_2 = \tfrac{1}{2}f''(a)$$

so that, substituting in Equation (2), the quadratic Taylor approximation to $f(x)$ (when x is close to a) is

Definition 1

$$f(x) \simeq f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2.$$

It is usually called the quadratic Taylor approximation to f *about* a. With this formula we can take any real function f and any element a in the domain of f, for which $f(a)$ and its first two derivatives are known, and calculate an approximation to the image value of any other element x close to a in the domain.

Exercise 1

Exercise 1
(2 minutes)

Use the quadratic Taylor approximation with $a = 1$ to evaluate $\exp(1.2)$ approximately, given that $\exp(1) = e = 2.72$ to 2 decimal places.

Compare the error in your result with the error of the tangent approximation for the same number, which was given in the solution to Exercise 14.1.1.2. ■

Exercise 2

Exercise 2
(3 minutes)

Use the quadratic Taylor approximation with $a = 0$ to estimate $\sin\left(\frac{\pi}{10}\right)$. In Exercise 14.1.1.1, the tangent approximation at $\frac{\pi}{6}$ gave us a 3% error in $\sin\left(\frac{\pi}{10}\right)$. What is the error in this new approximation? (The true value of $\sin\left(\frac{\pi}{10}\right)$ is 0.3090 to 4 decimal places.) ■

Solution 1 **Solution 1**

$$\exp(1) = \exp'(1) = \exp''(1) = 2.72$$

to 2 decimal places.

Therefore, using the quadratic Taylor approximation, we find:

$$\begin{aligned}\exp(1.2) &\simeq 2.72(1 + 0.2 + \tfrac{1}{2}(0.2)^2)\\ &= 2.72 \times 1.22\\ &= 3.32 \text{ to 2 decimal places}\end{aligned}$$

which agrees with the true value of $\exp(1.2)$ (given in the table in Exercise 14.1.1.2) to 2 decimal places. Thus, the magnitude of the error is less than or equal to $0.005 + 0.005 = 0.01$, so this approximation is at least 6 times as good as the tangent approximation to $\exp(1.2)$. ■

Solution 2 **Solution 2**

The quadratic Taylor approximation to the sine function about 0 gives

$$\begin{aligned}\sin\left(\frac{\pi}{10}\right) &\simeq 1 \times \left(\frac{\pi}{10}\right) + \frac{1}{2} \times 0 \times \left(\frac{\pi}{10}\right)^2\\ &= \frac{\pi}{10}\\ &= 0.3142 \text{ to 4 decimal places.}\end{aligned}$$

The quadratic Taylor approximation with $a = 0$ is in this case the same as the tangent approximation with $a = 0$, and is about twice as accurate as the tangent approximation with $a = \dfrac{\pi}{6}$ considered in Exercise 14.1.1.1 (the error is about 1.5% instead of 3%). ■

Main Text

An alternative derivation of the quadratic Taylor approximation is to take the limit as h approaches 0 in the Gregory–Newton formula for the quadratic that fits the given function f at the three tabular points a, $a + h$ and $a + 2h$. This formula was given in *Unit 4, Finite Differences*, section 4.3.2: it is:

$$\text{quadratic} = f(a) + \theta\Delta_h f(a) + \frac{\theta(\theta - 1)}{2}\Delta_h^2 f(a)$$

where $\theta = \dfrac{x - a}{h}$, $\Delta_h f(x) = f(x + h) - f(x)$, and $\Delta_h^2 = \Delta_h \circ \Delta_h$.

Substituting for θ in the formula on the right-hand side, we have

$$f(a) + \frac{\Delta_h f(a)}{h}(x - a) + \frac{\frac{1}{2}\Delta_h^2 f(a)}{h^2}(x - a)(x - a - h).$$

To obtain the Taylor approximation we take the limit as h approaches zero. Then, by the definition of a derivative (see *Unit 12, Differentiation I*, section 12.1.4) the coefficient of $(x - a)$ becomes $f'(a)$. The limit of the next coefficient, $\Delta_h^2 f(a)/h^2$, is not so easy to evaluate but the following argument helps: the definition of a derivative tells us that for small enough h we can approximately replace the operator $\dfrac{\Delta_h}{h}$ by the differentiation operator D, so it is plausible that we can also replace $\dfrac{(\Delta_h)^2}{h^2}$, that is,

$\frac{\Delta_h}{h} \circ \frac{\Delta_h}{h}$, by $D^2 = D \circ D$, in which case we would get

$$\lim_{h \to 0} \frac{\Delta_h^2 f(a)}{h^2} = D^2 f(a).$$

So we find that as h approaches 0 the limit of the Gregory–Newton formula gives

$$f(x) \simeq f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2$$

which is the same as the quadratic Taylor approximation we arrived at earlier.

Discussion

(The reason why this argument is not a proof is that all our definitions and theorems about approximations and limits refer to *numbers*. We have no definitions or theorems relating to the approximation of *operators* such as Δ_h and D, so that although we could write down $\Delta_h/h \simeq D$ or even $\lim_{h \to 0} \Delta_h/h = D$, these formulas would have no precise meanings, and we would be unable to deduce that $(\Delta_h/h)^2 \simeq D^2$ or $\lim_{h \to 0} (\Delta_h/h)^2 = D^2$. In fact it is possible to prove that these results do hold, provided the function f is, in a well-defined sense, *well-behaved* when x is close to a.)

Exercise 3

Exercise 3
(3 minutes)

Evaluate

$$\lim_{h \to 0} \frac{\Delta_h^2 f(a)}{h^2},$$

where $f(x) = x^3$ $(x \in R)$, and a is any given real number; compare the result with the value of $D^2 f(a)$ obtained from the rules of differentiation.

■

Solution 3

Solution 3

$$\frac{\Delta_h^2 f(a)}{h^2} = \frac{\Delta_h(\Delta_h f)(a)}{h^2}$$
$$= \frac{\Delta_h f(a+h) - \Delta_h f(a)}{h^2}$$
$$= \frac{f(a+2h) - f(a+h) - f(a+h) + f(a)}{h^2}.$$

With $f(x) = x^3$, this reduces to

$$\frac{\Delta_h^2 f(a)}{h^2} = \frac{6h^2a + 6h^3}{h^2} = 6a + 6h.$$

Thus, we have

$$\lim_{h \to 0} \frac{\Delta_h^2 f(a)}{h^2} = 6a \qquad \text{when } f(x) = x^3 \qquad (x \in R).$$

Differentiating f, we get $Df(x) = 3x^2$ and $D^2f(x) = 6x$, and so $D^2f(a) = 6a$, which is the same as $\lim_{h \to 0} \frac{\Delta_h^2 f(a)}{h^2}$ in this particular case. ■

14.1.5 The General Taylor Approximation

In the previous section we showed how the quadratic Taylor approximation gives, in general, a better approximation than the linear one (the tangent approximation); but for some purposes (such as the problem considered on television, the calculation of $\sin\left(\frac{\pi}{10}\right)$ to 7 places of decimals) even the quadratic approximation is not adequate. To look for even better approximations, it is natural to try the same method with a cubic polynomial, or one of even higher degree.

In this section we formulate the Taylor approximation that uses a polynomial of any degree, say the nth. By analogy with the method that worked for the quadratic polynomial, let us write the polynomial of degree n in the form:

$$p(x) = b_0 + b_1(x - a) + b_2(x - a)^2 + \cdots + b_n(x - a)^n$$

where $b_0, b_1, \ldots, b_n$ are numbers. (At this stage there is no reason to assume any connection between the numbers b_0, b_1 and b_2 in this section and the coefficients of the quadratic polynomial in the preceding section, but we shall see presently that they are in fact the same.) How do we determine the numbers $b_0, \ldots, b_n$? Since there are $n + 1$ of them, we need $n + 1$ conditions to fix them all. From the previous section we already have three conditions

$$p(a) = f(a)$$
$$p'(a) = f'(a)$$
$$p''(a) = f''(a)$$

where f is the function we are trying to approximate. It is natural to impose the remaining conditions by continuing the list:

$$p'''(a) = f'''(a)$$
$$p^{(4)}(a) = f^{(4)}(a)$$
$$\cdots$$
$$p^{(n)}(a) = f^{(n)}(a)$$

where $f^{(n)}(a)$ means the nth derivative of f at a. The complete list gives us exactly $n + 1$ conditions, and it is plausible to use these to determine the numbers $b_0, \ldots, b_n$ in the definition of p. The next two exercises deal with the determination of these numbers.

Exercise 1

Exercise 1
(3 minutes)

If c is a polynomial function defined by

$$c(x) = b_0 + b_1(x - a) + b_2(x - a)^2 + b_3(x - a)^3$$

and $c(a)$ and the first three derivatives $c'(a)$, $c''(a)$ and $c'''(a)$ are equal to $f(a)$, $f'(a)$, $f''(a)$ and $f'''(a)$ respectively, find b_0, b_1, b_2 and b_3, and hence write down a formula giving c in terms of f and its first three derivatives at a. ■

Exercise 2

Exercise 2
(3 minutes)

Guess the formula for the Taylor approximation by a polynomial of degree n, where n is any positive integer. (The answer is given in the text following Solution 1.) ■

Solution 1 **Solution 1**

We have

$$c(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3$$
$$c'(x) = b_1 + 2b_2(x-a) + 3b_3(x-a)^2$$
$$c''(x) = 2b_2 + 6b_3(x-a)$$
$$c'''(x) = 6b_3.$$

Therefore

$c(a) = b_0$ (and we are given that $c(a) = f(a)$)

$c'(a) = b_1$ (and we are given that $c'(a) = f'(a)$)

$c''(a) = 2b_2$ (and we are given that $c''(a) = f''(a)$)

$c'''(a) = 6b_3$ (and we are given that $c'''(a) = f'''(a)$).

Thus, $b_0 = f(a)$, $b_1 = f'(a)$, $b_2 = \frac{1}{2}f''(a)$, $b_3 = \frac{1}{6}f'''(a)$, and so the formula for c is

$$c(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + \tfrac{1}{6}f'''(a)(x-a)^3.$$

■

Main Text

The formula for the nth degree Taylor polynomial approximation can be calculated by writing it in the form

$$p(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n$$

and using the $n + 1$ conditions

$$p(a) = f(a), p'(a) = f'(a), \ldots, p^{(n)}(a) = f^{(n)}(a)$$

to determine the $n + 1$ numbers $b_0, b_1, \ldots, b_n$.

Definition 1

We thus obtain the Taylor approximation of degree n:

$$f(x) \simeq f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + \cdots$$
$$\cdots + \underbrace{\frac{1}{k!}f^{(k)}(a)(x-a)^k}_{\text{general term}} + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

or, in summation notation,

$$f(x) \simeq f(a) + \sum_{k=1}^{n} \frac{1}{k!}f^{(k)}(a)(x-a)^k.$$

This is usually referred to as Taylor's approximation to *f about a*. The factorials* in the denominator arise because the kth derived function of $x \longmapsto (x-a)^k$ is $x \longmapsto k!$.

Definition 2

The value of a for which this approximation is simplest is usually 0, and the resultant form of the Taylor approximation is common enough to have a special name: it is called the Maclaurin approximation. Its formula is

$$f(x) \simeq f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + \cdots$$
$$\cdots + \frac{1}{k!}f^{(k)}(0)x^k + \cdots + \frac{1}{n!}f^{(n)}(0)x^n.$$

The Maclaurin approximation is a Taylor approximation to f about 0.

In the television programme we obtained the Maclaurin approximation directly, instead of via the Taylor approximation, and we applied it to

* The symbol $k!$ denotes the product $1 \times 2 \times 3 \times \cdots \times k$, and is read "factorial k".

the case where f is the sine function. For this function the derivatives at 0 are:

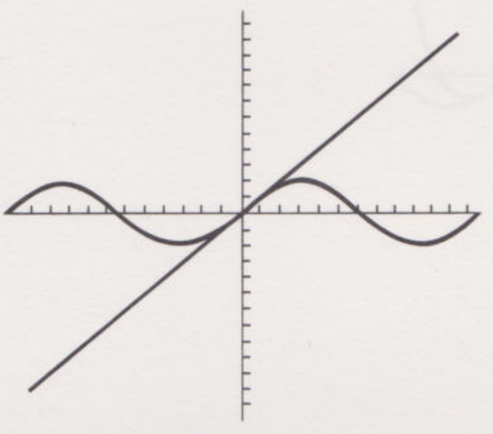
$$\begin{aligned}
f(0) &= \sin 0 = 0\\
f'(0) &= \cos 0 = 1\\
f''(0) &= -\sin 0 = 0\\
f'''(0) &= -\cos 0 = -1\\
f^{(4)}(0) &= \sin 0 = 0\\
f^{(5)}(0) &= \cos 0 = 1\\
&\ldots
\end{aligned}$$

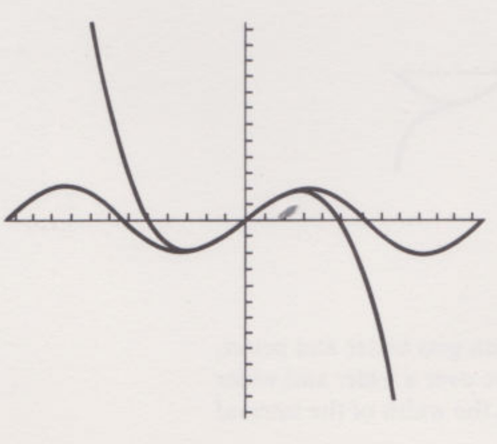
and the pattern $0, 1, 0, -1, 0, 1, 0, -1, \ldots$ goes on repeating itself; substituting these values into the Maclaurin approximation we get the successive approximations:

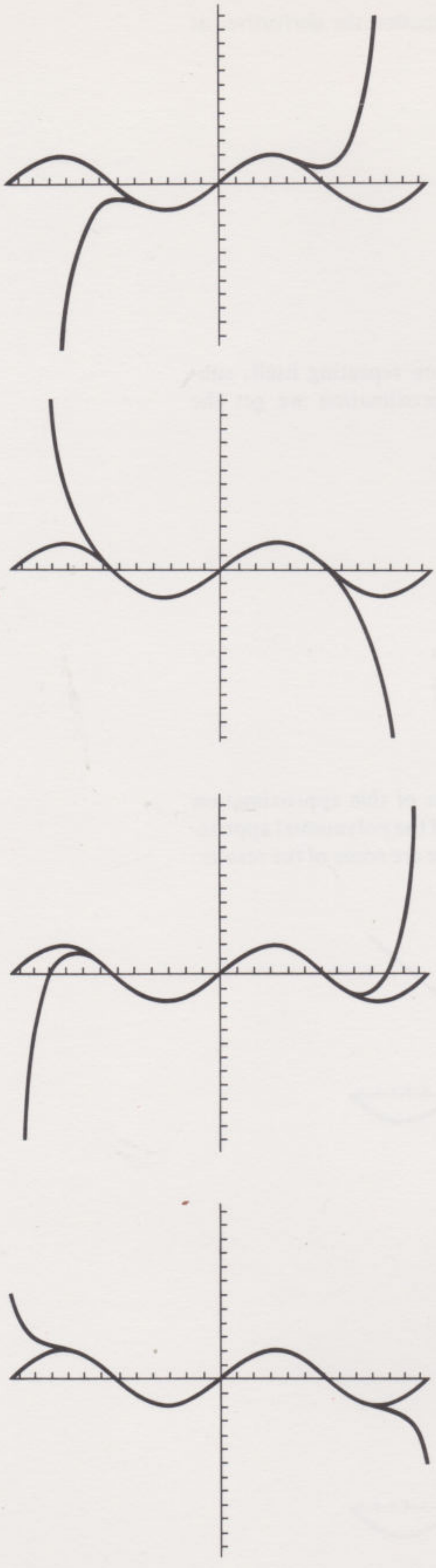
$$\begin{aligned}
&(n = 1 \text{ or } 2) \quad \sin x \simeq x\\
&(n = 3 \text{ or } 4) \quad \sin x \simeq x - \frac{x^3}{3!}\\
&(n = 5 \text{ or } 6) \quad \sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!}\\
&(n = 7 \text{ or } 8) \quad \sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\\
&\quad \ldots \qquad\qquad \ldots
\end{aligned}$$

and so on.

In the television programme we show two tests of this approximation method for $\sin x$. One is to compare the graphs of the polynomial approximations with the graph of the sine function. Here are some of the results:







As we make n larger and larger the approximation gets better and better, in the sense that the polynomial fits the sine curve over a wider and wider interval, and there seems to be no restriction on the width of the interval

if we take a polynomial of sufficiently high degree. But note that the diagrams are somewhat deceptive, since the lines must have some thickness in order to be visible, and all they really show is that the approximating curve gradually "follows" the sine curve as the degree of the approximating polynomial increases.

The second (and better) test used in the television programme is to calculate the first few approximations to $\sin\left(\frac{\pi}{10}\right)$. These are

$$n = 1 \quad \sin\left(\frac{\pi}{10}\right) \simeq \frac{\pi}{10} = 0.3141593$$

$$n = 3 \quad \sin\left(\frac{\pi}{10}\right) \simeq \frac{\pi}{10} - \frac{1}{3!}\left(\frac{\pi}{10}\right)^3 = 0.3089921$$

$$n = 5 \qquad = 0.3090176$$

$$n = 7 \qquad = 0.3090170.$$

If we continued with higher values of n the result would still be 0.3090170 to 7 significant figures. This number agrees with the measured value of 0.3090 ± 0.0001, to within the estimated error. This is a good demonstration (though not, of course, a proof) of the validity of Maclaurin's approximation for the sine function.

It is a remarkable fact that, knowing the images of the sine function and its derived functions at the single element 0, Maclaurin's formula gives us a method of investigating the images of *all* the real numbers under the sine function.

Exercise 3

Exercise 3
(3 minutes)

Find the general Maclaurin approximation to the exponential function, and calculate the first few Maclaurin approximations for $\exp(0.1)$ to 3 decimal places. Compare your results with the calculation of $\exp(0.1)$ directly from the definition, given in section 7.4.1 of *Unit 7, Sequences and Limits I*; the first 10 steps of that calculation are given in the table.

k	$\left(1 + \frac{0.1}{k}\right)^k$
1	1.1
2	1.1025
3	1.1034
4	1.1038
5	1.1041
6	1.1043
7	1.1044
8	1.1045
9	1.1046
10	1.1046

■

Exercise 4

Exercise 4
(2 minutes)

Find the general Maclaurin approximation to the cosine function, and calculate the first three distinct Maclaurin approximations for $\cos(0.3)$ to 3 decimal places. Compare your results with the true value, 0.9553. ■

In the examples considered so far, Maclaurin's approximation has been extremely successful; the following exercise shows that this is not always the case.

Exercise 5

Exercise 5
(5 minutes)

Find the general Maclaurin approximation to the function

$$x \longmapsto (1 - x)^s \qquad (x \in R, x \neq 1),$$

where s is any real number.

Do you recognize the approximation when s is a positive integer?

Calculate the first few Maclaurin approximations to $(1 - x)^{-1}$, where

(i) $x = 0.1$ (ii) $x = 10$. ■

Solution 3

Solution 3

Since the nth derivative of exp at x is $\exp(x)$ for all n, the general Maclaurin approximation of degree n for $\exp(x)$ is simply

$$\exp(x) \simeq \exp(0) + x\exp(0) + \tfrac{1}{2}x^2\exp(0) + \cdots + \frac{1}{n!}x^n\exp(0)$$

$$= 1 + x + \tfrac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n.$$

The first-degree approximation to $\exp(0.1)$ is

$$\exp(0.1) \simeq 1 + 0.1 = 1.1.$$

The second-degree approximation is

$$\exp(0.1) \simeq 1 + 0.1 + 0.005 = 1.105.$$

The third-degree approximation is again

$\exp(0.1) \simeq 1.105$ to 3 decimal places,

and the fourth and higher degree approximations also give 1.105. Thus the Maclaurin approximations for $\exp(0.1)$ converge much more quickly than the calculation directly from the definition — the second-degree approximation is already correct to 3 decimal places. ■

Solution 4

Solution 4

$D\cos(x) = -\sin x$ gives $D\cos(0) = 0$;

$D^2\cos(x) = -\cos x$ gives $D^2\cos(0) = -1$;

$D^3\cos(x) = \sin x$ gives $D^3\cos(0) = 0$;

$D^4\cos(x) = \cos x$ gives $D^4\cos(0) = 1$;

and so on.

Therefore, the Maclaurin approximation to the cosine function contains only *even* powers of x; so for any positive integer n the Maclaurin polynomial approximations of degrees $2n$ and $2n + 1$ are the same, and are given by

$$\cos x \simeq 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots + (-1)^n\frac{1}{(2n)!}x^{2n}.$$

The approximation of degree 0 (or 1) to $\cos(0.3)$ is therefore

$$\cos(0.3) \simeq 1.$$

The approximation of degree 2 (or 3) is

$$\cos(0.3) \simeq 1 - (\tfrac{1}{2} \times 0.09) = 0.955.$$

The approximation of degree 4 (or 5) is

$$\cos(0.3) \simeq 1 - (\tfrac{1}{2} \times 0.09) + (\tfrac{1}{24} \times 0.0081) = 0.9553$$

which agrees with the true value to 4 decimal places. ■

Solution 5

Solution 5

Let

$$f: x \longmapsto (1 - x)^s \qquad (x \in R,\ x \neq 1);$$

then

$$Df: x \longmapsto -s(1 - x)^{s-1} \qquad (x \in R,\ x \neq 1),$$

$$D^2f: x \longmapsto s(s - 1)(1 - x)^{s-2} \qquad (x \in R,\ x \neq 1),$$

and, for any $n \in Z^+$,

$$D^n f: x \longmapsto (-1)^n s(s-1)\ldots(s-n+1)(1-x)^{s-n}$$

$$(x \in R, x \neq 1).$$

The general Maclaurin approximation of degree n is

$$(1-x)^s \simeq 1 - sx + \frac{s(s-1)}{1 \times 2}x^2 - \frac{s(s-1)(s-2)}{1 \times 2 \times 3}x^3 + \cdots$$

$$\cdots + (-1)^n \frac{s(s-1) \times \cdots \times (s-n+1)}{1 \times 2 \times \cdots \times n}x^n.$$

When s is a positive integer, the sth coefficient is $(-1)^s$, and each coefficient thereafter has a zero in the numerator, and is therefore equal to zero. You should recognize the above polynomial as the binomial expansion for $(1-x)^s$, which gives the *exact* value of $(1-x)^s$ when s is a positive integer. (See RB9)

When $s = -1$, the situation is vastly different. The general Maclaurin approximation of degree n is now

$$(1-x)^{-1} \simeq 1 - (-1)x + \frac{(-1) \times (-2)}{1 \times 2}x^2$$

$$-\frac{(-1) \times (-2) \times (-3)}{1 \times 2 \times 3}x^3 + \cdots$$

$$\cdots + (-1)^n \frac{(-1) \times (-2) \times \cdots \times (-n)}{1 \times 2 \times \cdots \times n}x^n$$

$$= 1 + x + x^2 + \cdots + x^n.$$

In this case, since s is not a positive integer, there is no exact polynomial expression for $(1-x)^s$, and so there is no "final" polynomial in the sequence of Maclaurin approximations.

(i) When $x = 0.1$,

$$(1-x)^{-1} = \frac{1}{0.9} = \frac{10}{9} = 1.111\ldots$$

The first approximation is 1.1;

the second approximation is 1.11;

the third approximation is 1.111;

etc.

(ii) When $x = 10$,

$$(1-x)^{-1} = -\tfrac{1}{9} \qquad = -0.111\ldots$$

The first "approximation" is 11;

the second "approximation" is 111;

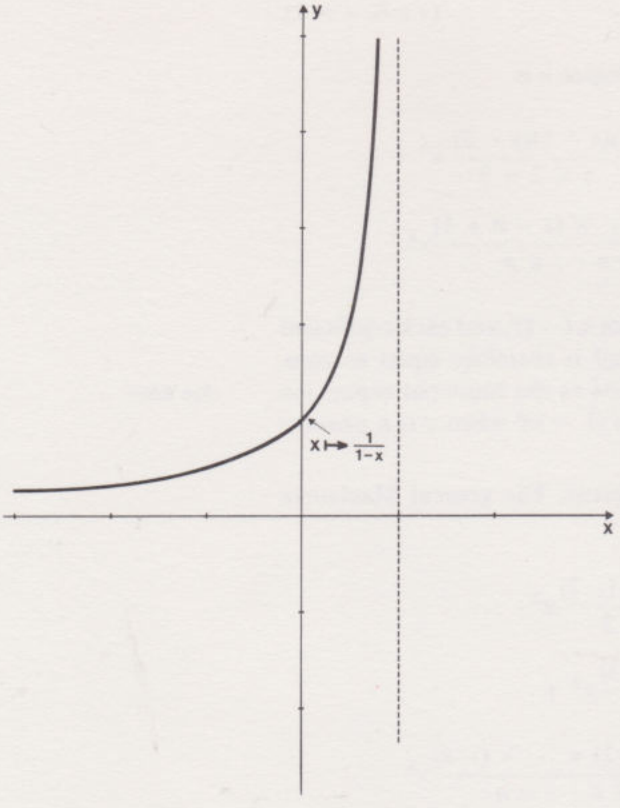
the third "approximation" is 1111;

etc. ■

Discussion **

In the last exercise the method was successful for $x = 0.1$, but for $x = 10$ the "approximations" bear no relation whatever to the correct value. Essentially the same thing can be shown by looking at the graphs of

the successive Maclaurin approximations to $(1 - x)^{-1}$; these are shown in the following diagram* and its overlays:



The graphs show that the nature of the approximation is not the same as in the case of $\sin x$: for $\sin x$ the interval over which the approximation is good gets wider as the degree of the polynomial gets higher; but for $(1 - x)^{-1}$ the interval of good approximation is always contained within the interval $[-1, 1]$.

These results show that the Taylor (Maclaurin) approximation method is quite temperamental: sometimes it is very effective, but on other occasions the approximations it produces are wide of the mark. The method is a very powerful one, but to be able to use it without getting into trouble one needs either very sound intuition or some theorems that will specify the situations in which the method is successful. In the next section of the text we shall leave the exploratory approach we have been using and look at the theory of the Taylor approximation method from a rigorous point of view.

* We have drawn only the part of the graph for which $1/(1 - x) > 0$; the dotted line is the line specified by $x = 1$.

The overlays are in the wallet on the inside of the back cover of this text.

14.2 INFINITE SERIES

14.2

14.2.1 Taylor's Theorem

14.2.1

Introduction

The main purpose of this section is to enable you to recognize the situations where the Taylor (or Maclaurin) approximation method works satisfactorily, so that you will know how to take advantage of the method without getting false results. To do this we shall follow the philosophy of *Unit 2, Errors and Accuracy*, and look for a bound on the error of the approximation.

To explain the principle of the method, we consider first how to estimate the error in the simplest of the Taylor polynomial approximations, the tangent approximation. The absolute error in any approximation is defined to be

$$(\text{absolute error}) = (\text{approximation}) - (\text{exact value})$$

(see *Unit 2, Errors and Accuracy*, section 2.1.1).

It is a little more convenient to work not with the error itself but with its negative, which is the correction that must be added to the approximation to cancel the error and thus yield the exact value:

$$(\text{correction}) = (\text{exact value}) - (\text{approximation}).$$

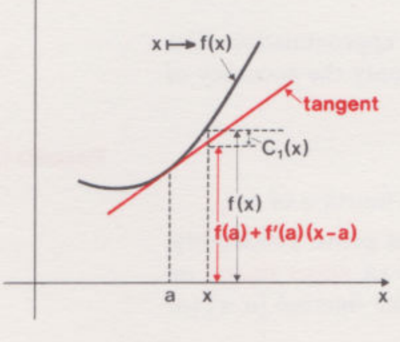
The error and the correction have the same magnitude (modulus), so that any bound on the magnitude of the correction is automatically an error bound too. For any given function f, let us denote the correction to the tangent approximation for $f(x)$ about some given point a by $C_1(x)$, the subscript 1 indicating that this refers to the Taylor approximation of degree one. The formula for the tangent approximation (which we found in section 14.1.1) is

$$f(x) \simeq f(a) + f'(a)(x - a),$$

and hence the correction is given by

$$C_1(x) = f(x) - (f(a) + f'(a)(x - a)). \qquad \text{Equation (1)}$$

Now $C_1(x)$ is the number we wish to estimate, but let us first get some idea of its size by trying some suitable approximations.



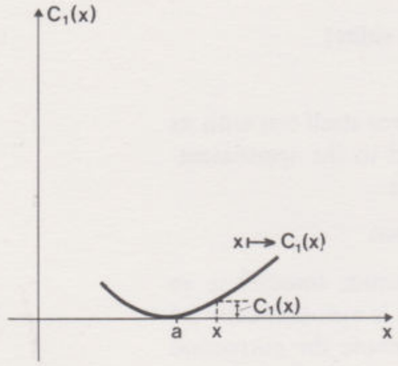
One way of getting an idea of the size of $C_1(x)$ is to replace $f(x)$ on the right of Equation (1) by a convenient approximation. What approximation would you suggest? The tangent approximation about a will not do, for that would give

$$C_1(x) \simeq (\text{tangent approx.}) - (\text{tangent approx.}) = 0$$

which is no help. But, by using the next Taylor polynomial for $f(x)$, we can get a useful estimate; it is

$$\begin{aligned} C_1(x) &\simeq (f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2) \\ &\quad -(f(a) + f'(a)(x-a)) \\ &= \tfrac{1}{2}f''(a)(x-a)^2. \end{aligned}$$

Thus, $C_1(x)$ is roughly proportional to the square of the distance $(x - a)$, and also to the second derivative of f at a. Both these facts can also be seen from the above figure, especially if it is redrawn to show how $C_1(x)$ depends on x.



Exercise 1

Exercise 1
(3 minutes)

Use the formula

$$C_1(x) \simeq \tfrac{1}{2}f''(a)(x-a)^2$$

to estimate (to 2 decimal places) the ~~error in the~~ correction to the tangent approximation at 1 to the exponential function for $x = 0.8, 0.9, 1.1, 1.2$, and compare with your calculated results for this error from Exercise 14.1.1.2. ■

Discussion **

The problem now is to convert the rough estimate for the correction to the tangent approximation about a,

$$C_1(x) \simeq \tfrac{1}{2}f''(a)(x-a)^2,$$

into a precise specification of the accuracy of this approximation. The above result suggests that it may be possible to specify the accuracy of the tangent approximation by a formula such as

$$|C_1(x)| \leqslant \tfrac{1}{2}B(x-a)^2$$ **Inequality (1)**

in which B is somehow related to the second derived function of f.

In fact, this method of specifying the accuracy does prove satisfactory. It can be shown that the result holds provided B is an upper bound on the magnitude of the second derivative of f over the interval $[a, x]$ (or $[x, a]$ if $x < a$); that is, *provided*

$$|f''(t)| \leqslant B \qquad (t \in [a, x]).$$ **Inequality (2)**

Inequalities (1) and (2) together constitute a statement of Taylor's Theorem for the tangent approximation. A proof is given in the Appendix.

Example 1

Example 1

As an illustration, let us apply Taylor's Theorem to the case already considered in Exercise 14.1.1.2 and Exercise 1, where f is the exponential

function and $a = 1$. In this case the tangent approximation is

$$\exp x \simeq \exp(1) + (x-1) \times \exp'(1)$$
$$= 2.7183 + (x-1) \times 2.7183,$$

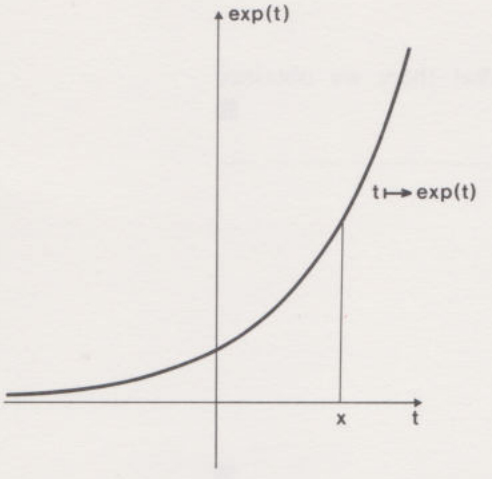
and Taylor's Theorem tells us that the correction satisfies the inequality

$$|C_1(x)| \leqslant \tfrac{1}{2}B(x-1)^2,$$

provided B satisfies the inequality

$$|\exp t| \leqslant B \qquad (t \in [1, x])$$

(since $\exp'' = \exp$).



Since $\exp t$ increases as t increases, its largest value for $t \in [1, x]$ is achieved when t is the largest number in the interval $[1, x]$, which is x if $x > 1$ and 1 if $x < 1$. Accordingly we can satisfy the last inequality by taking B to be the image under the exponential function of the largest number in the interval:

$$B = \begin{cases} \exp x & \text{if } x > 1 \\ e & \text{if } x < 1. \end{cases}$$

(We could take B larger than this if we wished, and still satisfy the required inequality, but this would weaken the bound given by the first inequality without gaining anything.) Thus Taylor's Theorem tells us that

$$\exp x \simeq 2.7183 + (x-1) \times 2.7183$$

with a correction of magnitude not exceeding

$$\begin{cases} \frac{1}{2}(\exp x) \times (x-1)^2 & \text{if } x > 1 \\ \frac{1}{2}e \times (x-1)^2 & \text{if } x < 1. \end{cases}$$

For example, if $x = 0.8$, Taylor's Theorem tells us that the magnitude of the correction cannot exceed

$$\tfrac{1}{2} \times 2.7183 \times (-0.2)^2 = 0.0544.$$

The actual correction is

$$\exp(0.8) - (2.7183 + (-0.2) \times 2.7183)$$
$$= 2.2255 - 2.1746$$
$$= 0.0509.$$

If $x = 1.2$, Taylor's Theorem tells us that the magnitude of the correction cannot exceed

$$\tfrac{1}{2} \times 3.3201 \times (0.2)^2 = 0.0664.$$

(continued on page 30)

Solution 1

Solution 1

$$\exp(1) = \exp'(1) = \exp''(1) = 2.72 \text{ (to 2 decimal places)}.$$

Therefore we have the following estimates (to 2 decimal places) for the ~~error in~~ correction to the tangent approximation at 1 for $x = 0.8, 0.9, 1.1, 1.2$:

$$C_1(0.8) \simeq \tfrac{1}{2} \times 2.72 \times (-0.2)^2$$
$$= 0.05$$
$$C_1(0.9) \simeq \tfrac{1}{2} \times 2.72 \times (-0.1)^2$$
$$= 0.01$$
$$C_1(1.1) \simeq 0.01$$
$$C_1(1.2) \simeq 0.05$$

which agrees with Exercise 14.1.1.2, except that there we obtained $C_1(1.2) = 0.06$ instead of 0.05. ■

(*continued from page 29*)

The actual correction is

$$\exp(1.2) - (2.7183 + (0.2) \times 2.7183)$$
$$= 3.3201 - 3.2620$$
$$= 0.0581.$$

Thus in both cases the theorem is verified. ■

Exercise 2

Exercise 2
(2 minutes)

Use Taylor's Theorem with $a = 0$ to obtain an absolute error bound for the approximation

$$\exp x \simeq 1 + x$$

for $x < 0$.
Deduce that

$$\exp(-0.2) \in [0.78, 0.82].$$ ■

Exercise 3

Exercise 3
(2 minutes)

Use Taylor's Theorem to obtain an absolute error bound for the tangent approximation about 0 to $\sin\left(\frac{\pi}{10}\right)$,

$$\sin\left(\frac{\pi}{10}\right) \simeq \frac{\pi}{10},$$

and compare it with the actual error. (The correct value of $\sin\left(\frac{\pi}{10}\right)$ is 0.3090 to 4 decimal places.) ■

*Exercise 4**

Exercise 4
(2 minutes)

If **p** denotes the proposition that Inequality (1) holds, and **q** denotes the proposition that Inequality (2) holds, which of the following propositions is logically equivalent to Taylor's Theorem?

(i) $\mathbf{p} \wedge \mathbf{q}$ (ii) $\mathbf{p} \vee \mathbf{q}$ (iii) $\mathbf{p} \Rightarrow \mathbf{q}$
(iv) $\mathbf{q} \Rightarrow \mathbf{p}$ (v) $\mathbf{p} \Leftrightarrow \mathbf{q}$ ■

* The symbols in this exercise were introduced in *Unit 11, Logic I.*

14.2.2 The General Taylor Theorem

14.2.2

Main Text

In this section we show how the method discussed in the preceding section can be generalized to give an upper bound on the correction to a Taylor approximation polynomial of general degree. This generalization makes it possible not only to estimate the error in any individual Taylor approximation, but also to see whether the sequence formed by the successive polynomial approximations of increasing degree to a given function value is convergent and has that function value as its limit; it also gives a new and powerful method of defining or specifying functions.

The Taylor approximation of degree n, obtained in section 14.1.5, is:

$$f(x) \simeq f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n.$$

The correction associated with this approximation is therefore

$$C_n(x) = f(x) - \left(f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n\right).$$

Just as in the case of the tangent approximation, we can get a rough approximation to $C_n(x)$ by using the next approximation for $f(x)$. We obtain this by replacing n by $n+1$ in the above Taylor approximation, and we find that

$$\begin{aligned} C_n(x) \simeq &\left(f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n\right. \\ &\left.+\frac{1}{(n+1)!}f^{(n+1)}(a)(x-a)^{n+1}\right) \\ &-\left(f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n\right). \end{aligned}$$

Thus

$$C_n(x) \simeq \frac{1}{(n+1)!}f^{(n+1)}(a)(x-a)^{n+1}.$$

This suggests that there may be a useful formula for the accuracy of the nth degree Taylor approximation of the form:

$$|C_n(x)| \leqslant \frac{1}{(n+1)!}B_{n+1}|(x-a)^{n+1}|$$

Inequality (1)

where B_{n+1} depends on $f^{(n+1)}$. As in the case of the tangent approximation ($n = 1$), it is possible to show that a sufficient condition for the above inequality to hold is

$$|f^{(n+1)}(t)| \leqslant B_{n+1} \qquad (t \in [a, x]),$$

Inequality (2)

where $f^{(n+1)}$ is continuous throughout the interval $[a, x]$, and where $[a, x]$ is to be interpreted as $[x, a]$ if $x < a$.

The statement that Inequality (2) implies Inequality (1) is the general form of Taylor's Theorem. The proof of this is given in the Appendix.

As an illustration of the use of Taylor's Theorem, let us apply it to the Maclaurin approximation,

$$\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!},$$

which is used in the television programme to calculate a number which appears to be $\sin\left(\frac{\pi}{10}\right)$ correct to 7 decimal places. We are now in a

(continued on page 33)

Solution 14.2.1.2

Solution 14.2.1.2

Taylor's Theorem tells us that the magnitude of the error, which equals the magnitude of the correction, cannot exceed

$$\tfrac{1}{2}Bx^2,$$

where $|\exp t| \leqslant B \; (t \in [x, 0])$.

Since $\exp t$ increases with t, and x is negative, the largest value of $\exp t$ for $t \in [x, 0]$ is $\exp 0 = 1$; so we may take $B = 1$. The magnitude of the error in the approximation,

$$\exp x \simeq 1 + x,$$

is therefore at most $\frac{1}{2}x^2$. In the particular case when $x = -0.2$ we have

$$\exp(-0.2) \simeq 1 - 0.2 = 0.8,$$

with an absolute error bound of $\frac{1}{2}(-0.2)^2 = 0.02$, from which it follows that

$$\exp(-0.2) \in [0.8 - 0.02, 0.8 + 0.02],$$

that is, $\exp(-0.2) \in [0.78, 0.82]$. ■

Solution 14.2.1.3

Solution 14.2.1.3

The tangent approximation we are using is

$$\begin{aligned} \sin x &\simeq \sin 0 + x \sin' 0 \\ &= x \end{aligned}$$

Taylor's Theorem tells us that the magnitude of the error cannot exceed

$$\tfrac{1}{2}Bx^2,$$

where $|\sin'' t| \leqslant B \; (t \in [0, x])$.

Since $\sin' = \cos$ and $\cos' = -\sin$, we have $\sin'' = -\sin$, and so the condition on B reduces to

$$|\sin t| \leqslant B \qquad (t \in [0, x]).$$

For the case when $x = \dfrac{\pi}{10}$, we require a number B such that

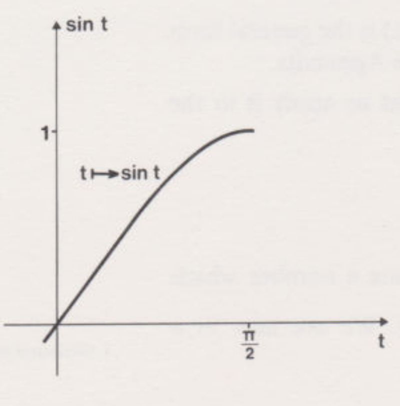
$$|\sin t| \leqslant B \qquad \left(t \in \left[0, \frac{\pi}{10}\right]\right).$$

For a quick estimate of the absolute error bound we may use the fact that $\sin t$ always lies in the range $[-1, 1]$, and take $B = 1$. We thus obtain the following value for the absolute error bound:

$$\frac{1}{2} \times 1 \times \left(\frac{\pi}{10}\right)^2 = \frac{\pi^2}{200} \simeq \frac{9.9}{200} = 0.05,$$

which is a perfectly satisfactory answer to the question.

Alternatively, we can do a little more work and get the "best" (that is, the smallest possible) value of B.



Since $\sin t$ increases with t in the interval $\left[0, \frac{\pi}{2}\right]$, its largest value in $\left[0, \frac{\pi}{10}\right]$ is $\sin \frac{\pi}{10} = 0.3090$. This gives the absolute error bound:

$$\tfrac{1}{2} \times 0.3090 \times \left(\frac{\pi}{10}\right)^2 \simeq 0.3090 \times 0.05 = 0.015.$$

(This is, of course, only of theoretical interest. Here we are discussing the accuracy of Taylor's approximation, but in a practical case we might want to *calculate* $\sin\left(\frac{\pi}{10}\right)$ *using* Taylor's approximation, and then we could not use our calculated value of $\sin\left(\frac{\pi}{10}\right)$ to obtain an error bound!)

The magnitude of the actual error is

$$\left|\sin\left(\frac{\pi}{10}\right) - \frac{\pi}{10}\right| = |0.3142 - 0.3090| = 0.0052.$$

Taylor's Theorem over-estimates the error by a factor of about 3 when the "best" value of B is used. ■

Solution 14.2.1.4 **Solution 14.2.1.4**

The statement of Taylor's Theorem given in the text is that Inequality (1) holds provided Inequality (2) holds; that is, if (2) holds, then (1) holds. In the notation used in the question, this statement is "if **q**, then **p**", and so the corresponding proposition is

(iv) $\mathbf{q} \Rightarrow \mathbf{p}$. ■

(*continued from page 31*)

position to prove that this number really is $\sin\left(\frac{\pi}{10}\right)$ correct to 7 decimal places. The correction we are interested in here is

$$C_7(x) = \sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right),$$

and according to Taylor's Theorem its magnitude has the upper bound

$$|C_7(x)| \leqslant \frac{1}{8!} B_8 |x|^8$$

where B_8 is such that

$$|\sin^{(8)} t| \leqslant B_8 \qquad (t \in [0, x]),$$

or, since the 8th derived function of sin is sin itself,

$$|\sin t| \leqslant B_8 \qquad (t \in [0, x]).$$

As in the solution to Exercise 14.2.1.3, it is not essential to find the "best" (that is, the smallest possible) value of B_8; any value satisfying the last inequality will do. Since we know that $|\sin t| \leqslant 1$ for all $t \in R$, it is convenient to take $B_8 = 1$. (We can always come back later and look for the best possible value of B_8 if the bound we get using $B_8 = 1$ turns out to be too weak.)

Substituting $B_8 = 1$ into the inequality for $|C_7(x)|$ gives:

$$|C_7(x)| \leqslant \frac{1}{8!}|x|^8 = \frac{|x|^8}{40\,320}.$$

Thus, if the 7th degree Taylor polynomial is used to calculate an approximate value for $\sin\left(\frac{\pi}{10}\right)$, the magnitude of the error is at most

$$\frac{\left(\frac{\pi}{10}\right)^8}{40\,320} < \frac{10^{-4}}{4 \times 10^4} \qquad (\text{since } \pi^2 = 9.9 < 10)$$
$$= \tfrac{1}{4} \times 10^{-8}.$$

Therefore, working to 8 decimal places, the error in the Taylor approximation is less than the possible error of $\frac{1}{2} \times 10^{-8}$ introduced by each arithmetical operation, and can be safely ignored when we have rounded-off the final answer to 7 decimal places.

The astonishing thing about this calculation is not the 7-figure accuracy, but the fact that we can rigorously estimate the error in our calculation of $\sin\left(\frac{\pi}{10}\right)$ without knowing in advance anything about the value of $\sin\left(\frac{\pi}{10}\right)$. We only used the first 7 derivatives of the sine function at 0 and the fact that the magnitude of the 8th derivative can never exceed 1.

Exercise 1

Exercise 1
(2 minutes)

In Exercise 14.1.5.3 we evaluated $\exp(0.1)$ using various Taylor polynomials about 0. In particular, the second-degree polynomial gave

$$\exp(0.1) \simeq 1 + 0.1 + \tfrac{1}{2} \times (0.1)^2 = 1.105.$$

Use Taylor's Theorem to find an absolute error bound for this approximation. You may assume that $\exp(0.1) < 2$. ■

14.2.3 Convergence of an Approximation Sequence

14.2.3

Discussion

The fact that we were able to obtain such an accurate approximation to $\sin\left(\frac{\pi}{10}\right)$ using Maclaurin polynomials suggests that we may be able to get any accuracy we please if we use polynomials of sufficiently high degree.

Is it really possible to get any accuracy we please? To answer this question, we examine the absolute error bound, $C_n\left(\frac{\pi}{10}\right)$; the question is whether we can make $C_n\left(\frac{\pi}{10}\right)$ as small as we please by making n large enough — that is, whether

$$\lim_{n\,\text{large}} C_n\left(\frac{\pi}{10}\right) = 0,$$

where

$$C_n\left(\cancel{\frac{\pi}{10}}\right) = \sin\left(\cancel{\frac{\pi}{10}}\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \frac{x^n}{n!} \text{ or } \frac{x^{n-1}}{(n-1)!}\right).$$

(The last term in the polynomial has magnitude $\frac{x^n}{n!}$ if n is odd, but $\frac{x^{n-1}}{(n-1)!}$ if n is even.)

Taylor's Theorem tells us that

$$|C_n(x)| \leqslant \frac{1}{(n+1)!} B|x|^{n+1},$$

provided B satisfies

$$|\sin^{(n)} t| \leqslant B \qquad (t \in [0, x]).$$

Whatever positive integer, n, we choose, $\sin^{(n)} t$ is one of $\sin t$, $\cos t$, $-\sin t$ and $-\cos t$, and so $|\sin^{(n)} t| \leqslant 1$ for all $t \in R$; so we can safely choose $B = 1$. Then we have

$$\left|C_n\left(\frac{\pi}{10}\right)\right| \leqslant \frac{1}{(n+1)!}\left(\frac{\pi}{10}\right)^{n+1}$$

$$\leqslant \left(\frac{1}{2}\right)^{n+1}, \quad \text{since} \quad \frac{\pi}{10} < \frac{1}{2}.$$

Consequently, we see that we *can* ensure that $C_n\left(\frac{\pi}{10}\right)$ is as small as we please by making n large enough. For example, to ensure that $\left|C_n\left(\frac{\pi}{10}\right)\right| < 2^{-1000}$, it is sufficient to take $n = 999$. In other words, we have shown that

$$\lim_{n \text{ large}} C_n\left(\frac{\pi}{10}\right) = 0.$$

Therefore, whatever accuracy is chosen, it is possible to specify n such that the Maclaurin polynomial of degree n gives $\sin\left(\frac{\pi}{10}\right)$ to the required accuracy.

A similar analysis can be carried out for the Maclaurin approximation to $\sin x$ for *any* x, and gives the same result: for any real number x and any stated accuracy, an integer n can be specified such that the Maclaurin polynomial of degree n gives $\sin x$ to the required accuracy. This fact (which we shall not prove here, though the proof is not difficult) is illustrated in the film in the television programme, which shows how the interval over which the approximating curve follows the sine curve expands as we include more and more terms in the Maclaurin polynomial. Similar results apply for other functions, for example, the cosine and the exponential functions. There are, however, functions for which the method only works for an interval of finite width in the domain of the function and others for which it does not work at all. We look at some of these in the final section of this text.

Exercise 1

Exercise 1
(5 minutes)

Write down the Maclaurin polynomial approximations of degrees 1, 2, 3 and 4 for $\cos x$. Use Taylor's Theorem to find an integer n such that the Maclaurin polynomial of degree n for $\cos x$ gives an approximation for $\cos(2)$ that is accurate to 2 decimal places, and write down the Maclaurin polynomial of this degree. (If you find this part very difficult, Solution 14.1.5.4 may be of some help.) The table of factorials may also be useful:

n	$n!$
1	1
2	2
3	6
4	24
5	120
6	720
7	5 040
8	40 320
9	362 880
10	3 628 800

■

Solution 14.2.2.1

Solution 14.2.2.1

The correction to the Taylor polynomial approximation we are considering is

$$C_2(x) = \exp x - (1 + x + \tfrac{1}{2}x^2),$$

and Taylor's Theorem tells us that

$$|C_2(x)| \leqslant \frac{1}{3!}B_3|x|^3$$

where

$$|\exp''' t| \leqslant B_3 \qquad (t \in [0, x]).$$

Since all the derived functions of exp are also exp, and the value of x we are interested in is 0.1, the condition on B_3 can be written:

$$|\exp t| \leqslant B_3 \qquad (t \in [0, 0.1]).$$

Since $\exp t$ increases with t, its maximum value for t in $[0, 0.1]$ is $\exp(0.1)$, which we know to be less than 2. So we may take $B_3 = 2$; then Taylor's Theorem gives

$$|C_2(0.1)| \leqslant \tfrac{1}{6} \times 2 \times (0.1)^3 = \tfrac{1}{3} \times 10^{-3},$$

and so the absolute error bound we are seeking is

$$\tfrac{1}{3} \times 10^{-3}.$$

This shows, incidentally, that

$$\exp(0.1) = 1.105 \text{ to 3 decimal places,}$$

since 1.105 is the exact value of the quadratic Taylor approximation. ■

Solution 1

Solution 1

The required Maclaurin polynomials are:

degree 1: $\cos x \simeq 1$

degrees 2, 3: $\cos x \simeq 1 - \tfrac{1}{2}x^2$

degree 4: $\cos x \simeq 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4$

Taylor's Theorem tells us that

$$|C_n(x)| \leqslant \frac{1}{(n+1)!}B_{n+1}|x|^{n+1},$$

provided $|\cos^{(n+1)} t| \leqslant B_{n+1}$ $(t \in [0, x])$.

Since all the derived functions of cos are $\pm\cos$ or $\pm\sin$, B_{n+1} can be taken as 1 for all n; so, with $x = 2$, our problem is to find an n such that

$$\frac{1}{(n+1)!}2^{n+1} \leqslant \tfrac{1}{2} \times 10^{-2}, \quad \text{to ensure that } |C_n(x)| \leqslant \tfrac{1}{2} \times 10^{-2}.$$

Trying successive values of n, and using the table of factorials, we obtain:

$$\frac{1}{7!}2^7 = \frac{128}{5\,040} \simeq \frac{1}{50} > \frac{1}{2} \times 10^{-2}$$

$$\frac{1}{8!}2^8 = \frac{256}{40\,320} \simeq \frac{1}{160} > \frac{1}{2} \times 10^{-2}$$

$$\frac{1}{9!}2^9 = \frac{512}{362\,880} < \frac{600}{360\,000} = \frac{1}{600} < \frac{1}{2} \times 10^{-2}$$

Thus the conditions of the problem are satisfied with $n = 8$. You may have chosen a value of n larger than 8. This is also correct: it gives an even smaller absolute error bound.

The Maclaurin polynomial approximation of degree 8 for $\cos x$ is

$$\cos x \simeq 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8. \quad \blacksquare$$

14.2.4 Infinite Series

14.2.4

Main Text

So far we have shown how to obtain various polynomial approximations to the image of a given function, sin say, for a given element, x say, in its domain:

$$\sin x \simeq x$$

$$\sin x \simeq x - \frac{x^3}{3!}$$

$$\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

etc.

We have shown that, in favourable cases (of which this example is one) the sequence of successive approximations thus obtained converges and has the exact image value as its limit. This sequence of successive approximations differs a little from the ones we considered in *Unit 7, Sequences and Limits I,* in that each new element of the sequence is calculated not from a recurrence formula, but by adding a term, such as $\frac{-x^3}{3!}$ or $\frac{x^5}{5!}$ to the preceding approximation. The successive terms that we may add also form an infinite sequence:

$$x, \quad \frac{-x^3}{3!}, \quad \frac{x^5}{5!}, \quad \frac{-x^7}{7!}, \ldots$$

To calculate one of the polynomial approximations to $\sin x$, we choose a positive integer n, and add up the first n consecutive members of this sequence. The more consecutive members we add in, the better is the approximation to $\sin x$. This is usually represented by writing

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Equation (1)

The expression on the right-hand side of this equation is called an *infinite series.* The three dots are used to indicate that the expression does not terminate.

The successive approximations:

$$x,$$

$$x - \frac{x^3}{3!},$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

etc.

are called the *partial sums* of the infinite series. Here the sequence of partial sums converges to a limit, namely $\sin x$; this limit is called the (total) *sum* of the infinite series.

It is very important to understand just what we mean by an infinite series. We give these important definitions formally:

An infinite series is an expression of the form **Definition 1**

$$a_1 + a_2 + a_3 + \cdots$$

The partial sums of the infinite series are the sums: **Definition 2**

$$S_k = a_1 + a_2 + \cdots + a_k \qquad (k = 1, 2, 3, \ldots).$$

If the sequence of partial sums,

$$S_1, S_2, S_3, \ldots$$

converges to a limit S, then we say that the series converges (or is convergent) to the sum S, and we write **Definition 3**

$$S = a_1 + a_2 + a_3 + \cdots.$$ **Notation 1**

If the sequence of partial sums does not converge, then we say that the series diverges (or is divergent): we cannot find a sum for it. **Definition 4**

It is important to note the difference between the infinite series

$$a_1 + a_2 + a_3 + \cdots$$

and the infinite sequence

$$a_1, a_2, a_3, \ldots$$

An *infinite series* can be thought of as a way of specifying an *infinite sequence of addition sums.*

Example 1 **Example 1**

You may have met the formula for the sum of k terms of a geometric progression,

$$a + ar + ar^2 + \cdots + ar^{k-1} = a\left(\frac{1 - r^k}{1 - r}\right) \qquad (r \in R, r \neq 1).$$

This is the kth partial sum, S_k, of the infinite series

$$a + ar + ar^2 + \cdots.$$

This series is called the infinite geometric series; the number r is called the common ratio. **Definition 5** (See RB 8)

As an example, let us take $a = 1$ and $r = \frac{1}{2}$; then we have:

$$\begin{aligned} 1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots + 2^{-(k-1)} &= \frac{1 - 2^{-k}}{\frac{1}{2}} \\ &= 2 - 2^{-(k-1)}. \end{aligned}$$

Thus the sequence $S_1, S_2, S_3, \ldots$ is now

$$2 - 1, 2 - \tfrac{1}{2}, 2 - \tfrac{1}{4}, \ldots$$

which converges to 2, so we can write

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots = 2.$$ ■

Exercise 1 **Exercise 1** (2 minutes)

Obtain a formula for the kth partial sum of

$$1 - 1 + 1 - 1 + 1 \ldots$$

Does the series converge or diverge? ■

Exercise 2

Exercise 2
(3 minutes)

For what values of r can we define a sum for the infinite geometric series

$$1 + r + r^2 + r^3 + \cdots$$

and what is the formula for the sum in each case? ■

Exercise 3

Exercise 3
(3 minutes)

For a given function f and given numbers x and a, if the corrections $C_k(x)$ satisfy

$$\lim_{k \text{ large}} C_k(x) = 0,$$

what can we conclude about the infinite series:

$$f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2 + \cdots?$$

What can we conclude if the corrections do *not* satisfy the above condition? ■

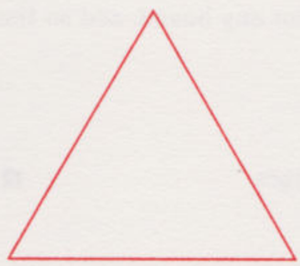
Exercise 4

Exercise 4
(5 minutes)

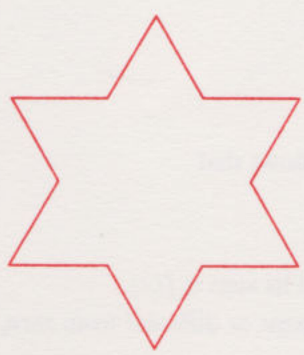
The "snowflake curve" is the limit of a sequence of polygons formed as follows:

a new line segment at each stage has length $\frac{1}{3}$ of that in the previous stage.

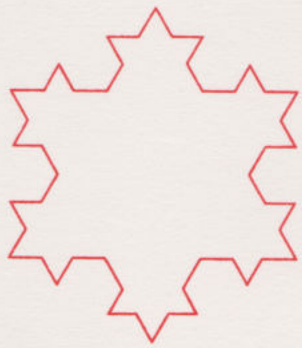
first polygon:



second polygon:



third polygon:



(*continued on page 41*)

Solution 1

Solution 1

$$S_k = \begin{cases} 0 \text{ if } k \text{ is even} \\ 1 \text{ if } k \text{ is odd} \end{cases}$$

and, since the sequence of partial sums 1, 0, 1, 0, ... diverges, the series also diverges. ■

Solution 2

Solution 2

The formula given in Example 1 gives, for the kth partial sum,

$$S_k = 1 + r + r^2 + \cdots + r^{k-1} = \frac{1 - r^k}{1 - r} \qquad (r \in R, r \neq 1).$$

We are interested in the behaviour of this expression for large k. This depends on the value of r, and so there are several cases to consider.

(i) If $|r| < 1$, then $\lim_{k \text{ large}} r^k = 0$, and so

$$\lim_{k \text{ large}} \frac{1 - r^k}{1 - r} = \frac{1}{1 - r}.$$

In this case the series converges and its sum is $\frac{1}{1 - r}$.

(ii) If $r = 1$, then the formula for S_k does not apply; we see that $S_k = k$, and so the series diverges.

(iii) If $|r| > 1$, then $|r^k|$ increases with k, without any bound, and so the series diverges.

(iv) If $r = -1$, we have the series

$$1 - 1 + 1 - 1 + 1 \ldots$$

which, as we have seen in Solution 1, diverges. ■

Solution 3

Solution 3

The kth partial sum, S_k, of the infinite series is the $(k - 1)$th degree Taylor approximation to $f(x)$ about $x = a$. Thus we have

$$f(x) \simeq S_k$$

with correction $C_k(x)$; or, in other words,

$$f(x) = S_k + C_k(x).$$

Thus, if we are given that $\lim_{k \text{ large}} C_k(x) = 0$, it follows that

$$f(x) = \lim_{k \text{ large}} S_k.$$

Consequently, the infinite series converges and its sum is $f(x)$.

In the cases when $\lim_{k \text{ large}} C_k(x)$ is either non-existent or different from zero, the conclusion is that the series either diverges or converges to a limit different from $f(x)$. ■

and so on. At each stage, every line segment

(*continued from page 39*)

in the old figure is changed to

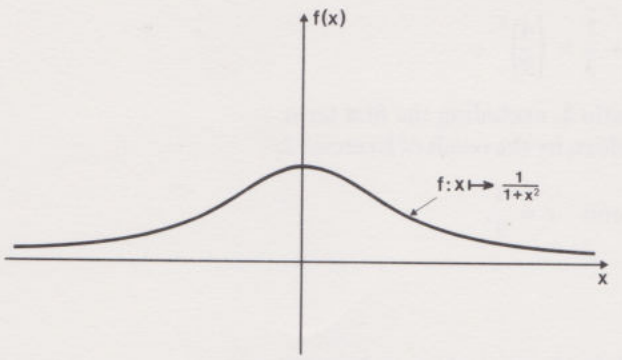
in the new figure in such a way as always to increase the enclosed area. Calculate the limiting area enclosed, taking the area of the triangle (first polygon) as 1 unit. What can be said about the limiting length of the perimeter? (All angles are 60° or 120°.) ■

Exercise 5

Exercise 5
(3 minutes)

Consider the function

$$f : x \longmapsto \frac{1}{1 + x^2} \qquad (x \in R).$$



By considering the geometric series

$$1 - x^2 + x^4 - x^6 + \cdots$$

obtain a sequence of polynomial approximations for $\frac{1}{1 + x^2}$. (These approximations are the Maclaurin polynomials for f.) Use the results of Exercise 2 to find the set of values of x for which this sequence converges to $\frac{1}{1 + x^2}$. ■

Exercise 6

Exercise 6
(3 minutes)

In *Unit 13, Integration II*, section 13.2.6, we mentioned the possibility of evaluating π using the formula

$$\frac{\pi}{4} = \int_0^1 x \longmapsto \frac{1}{1 + x^2}.$$

Use the approximations obtained in the preceding exercise to get a sequence of successive approximations to $\frac{\pi}{4}$. Assuming that this sequence really does converge to the limit $\frac{\pi}{4}$, write down an infinite series whose sum is $\frac{\pi}{4}$. ■

Solution 4

Solution 4

Let a_1 be the area of the triangle ($=1$), and for each $n = 2, 3, \ldots$ let a_n be the area added at the $(n-1)$th stage. This area is added in the form of b_n congruent triangles, each having one-third of the linear dimensions, and therefore $\frac{1}{9}$ of the area, of those added at the previous stage. Thus, the area of each triangle added at the $(n-1)$th stage is $(\frac{1}{9})^{n-1}$, and so

$$a_n = b_n \times 9^{-(n-1)} \qquad (n = 2, 3, \ldots).$$

Now b_n, the number of triangles added at the $(n-1)$th stage is equal to the number of line segments created at the previous stage. At the first stage, the number of line segments created is 3, and this is multiplied by 4 each stage. Thus,

$$b_n = 3 \times 4^{(n-2)} \qquad (n = 2, 3, \ldots).$$

This gives, when substituted in the previous equation,

$$a_n = \frac{1}{3} \times \left(\frac{4}{9}\right)^{(n-2)} \qquad (n = 2, 3, \ldots).$$

The total area at the $(n-1)$th stage is

$$a_1 + a_2 + \cdots + a_n,$$

and so the limiting area can be expressed as the infinite series:

$$1 + \frac{1}{3} + \frac{1}{3} \times \frac{4}{9} + \frac{1}{3} \times \left(\frac{4}{9}\right)^2 + \cdots + \frac{1}{3} \times \left(\frac{4}{9}\right)^k + \cdots$$

which is a geometric series with common ratio $\frac{4}{9}$, excluding the first term, which is an "odd man out". The sum is therefore, by the result of Exercise 2,

$$S = 1 + \frac{a}{1-r}, \quad \text{where} \quad a = \frac{1}{3} \quad \text{and} \quad r = \frac{4}{9},$$

$$= 1\tfrac{3}{5},$$

which is thus the limiting area.

The length of the perimeter of the snowflake does not fare so happily, however. At each stage the number of line segments is multiplied by 4, and the length of each line segment is divided by 3, so that the *total* length of the perimeter is *multiplied* by $\frac{4}{3}$ at each stage. Thus the lengths of the successive polygons are

$$1, \frac{4}{3}, \left(\frac{4}{3}\right)^2, \left(\frac{4}{3}\right)^3, \ldots$$

This sequence diverges, and in fact the length increases beyond all bounds. There is no "limiting length". ■

Solution 5

Solution 5

The series

$$1 - x^2 + x^4 - x^6 + \cdots$$

is a geometric series with common ratio $(-x^2)$, and therefore has the sum

$$\frac{1}{1-(-x^2)} = \frac{1}{1+x^2},$$

whenever $|-x^2| = x^2 < 1$; that is, whenever $|x| < 1$. The series diverges if $|x| \geqslant 1$. The partial sums of this series are the polynomials

$$1 - x^2, \qquad 1 - x^2 + x^4, \qquad 1 - x^2 + x^4 - x^6, \ldots$$

which accordingly form a convergent sequence of approximations to $\dfrac{1}{1+x^2}$ if $|x| < 1$, but not otherwise. ■

Solution 6

Solution 6

Since the sequence of successive approximations obtained in the preceding exercise converges to $\frac{1}{1+x^2}$ for all x in the interval of integration, with the single exception of the end-point 1, it is reasonable to guess that by successively substituting these polynomials for $\frac{1}{1+x^2}$ in the integral we shall obtain a sequence of successive approximations to $\frac{\pi}{4}$. This sequence is:

$$\int_0^1 (x \longmapsto 1 - x^2) = \left[x \longmapsto x - \frac{1}{3}x^3\right]_0^1 = 1 - \frac{1}{3}$$

$$\int_0^1 (x \longmapsto 1 - x^2 + x^4) = \left[x \longmapsto x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{5}$$

and so on: the nth approximation in the sequence is

$$\frac{\pi}{4} \simeq 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^n \frac{1}{2n+1}.$$

The corresponding infinite series is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

(This infinite series is mentioned in the radio programme.) ■

Discussion

The infinite series notation provides a convenient way of summarizing the type of result obtained in the second part of this text. For example, by writing

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \qquad (x \in R)$$

we can concisely express a statement that would otherwise look something like this:

"the correction $C_n(x)$ to the nth degree Maclaurin approximation

$$\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \cancel{+ \frac{x^m}{m!}} \; (-1)^{\frac{m-1}{2}} \frac{x^m}{m!}$$

where $m = n$ if n is odd, and $m = n - 1$ if n is even, satisfies

$$\lim_{n \text{ large}} C_n(x) = 0$$

for all $x \in R$".

Similarly, by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad (x \in R, |x| < 1)$$

we paraphrase the statement:

"if $|x| < 1$, then the sum $S_n(x)$ of the geometric series

$$1 + x + x^2 + \cdots + x^{n-1}$$

satisfies

$$\lim_{n \text{ large}} S_n(x) = \frac{1}{1-x}\text{''}.$$

To conclude this section, we summarize (for reference) a number of useful formulas of this kind. Some of them embody results already obtained in this unit, and some are new.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (x \in R);$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (x \in R);$$

$$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad (x \in R);$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (x \in R, |x| < 1)$$

$$(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!}x^3 + \cdots$$
$$(x \in R, |x| < 1),$$

where α is any real number. If α is a positive integer or zero, then all the terms of the last series after the $(\alpha + 1)$th are 0, so that the series reduces to a polynomial of degree α, and for these values of α the formula holds for all real x and not just for those satisfying $|x| < 1$.

Exercise 7

Exercise 7
(2 minutes)

Transcribe the following formula into a statement about Taylor polynomials:

$$\frac{1}{\sqrt{x}} = 1 - \frac{1}{2}(x - 1) + \frac{1 \times 3}{2 \times 4}(x - 1)^2 - \frac{1 \times 3 \times 5}{2 \times 4 \times 6}(x - 1)^3 + \cdots$$
$$+(-1)^n \frac{1 \times 3 \times 5 \times \cdots \times (2n - 1)}{2 \times 4 \times 6 \times \cdots \times (2n)}(x - 1)^n + \cdots$$
$$(x \in R^+ \text{ and } x < 2) \quad \blacksquare$$

14.2.5 Appendix (Not Part of the Course)

14.2.5 Appendix

Demonstration of Taylor's Theorem

We wish to show that if B satisfies the inequality

$$|f^{(n+1)}(t)| \leqslant B \qquad (t \in [a, x]),$$

Inequality (1)

then

$$|C_n(x)| \leqslant \frac{1}{(n+1)!} B|x-a|^{n+1}$$

Inequality (2)

where

$$C_n(x) = f(x) - \left[f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a)\right]$$

Inequality (1) is intended to imply that the $(n+1)$th derivative of f exists at all points in $[a, x]$. We define the function

$$C_n : t \longmapsto f(t) - \left[f(a) + (t-a)f'(a) + \frac{1}{2}(t-a)^2 f''(a) + \cdots + \frac{1}{n!}(t-a)^n f^{(n)}(a)\right] \qquad (t \in [a, x]).$$

This definition is consistent with the definition of $C_n(x)$ already given, and it makes sense for all points in the domain of f, since we have stipulated that f is $n+1$ times differentiable at all points in this domain.

We shall estimate $C_n(x)$ by estimating its $(n+1)$th derivative and then integrating $n+1$ times. Differentiating the function C_n, we obtain:

$$C_n'(t) = f'(t) - \left[f'(a) + (t-a)f''(a) + \cdots + \frac{1}{(n-1)!}(t-a)^{n-1} f^{(n)}(a)\right]$$

$$C_n''(t) = f''(t) - \left[f''(a) + \cdots + \frac{1}{(n-2)!}(t-a)^{n-2} f^{(n)}(a)\right]$$

$$\cdots$$

$$C_n^{(n)}(t) = f^{(n)}(t) - f^{(n)}(a)$$

$$C_n^{(n+1)}(t) = f^{(n+1)}(t)$$

where $t \in [a, x]$ in each case.

Combining the last equation with Inequality (1), we obtain the estimate

$$|C_n^{(n+1)}(t)| \leqslant B \qquad (t \in [a, x]).$$

Inequality (3)

We can use this information to estimate $C_n(x)$ itself, by a succession of $n+1$ integrations.

For simplicity we confine detailed discussion to the case where $x > a$, and to the upper bound on $C_n^{(n+1)}(t)$ implied by Inequality (3). The other cases can be treated similarly. The upper bound on $C_n^{(n+1)}(t)$ given by Inequality (3) is

$$C_n^{(n+1)}(t) \leqslant B \qquad (t \in [a, x]).$$

Integrating from a to s, where $s \in [a, x]$, gives (see diagram):

$$\int_a^s C_n^{(n+1)} \leqslant \int_a^s (t \longmapsto B)$$

(*continued on page 46*)

Solution 14.2.4.7

Solution 14.2.4.7

The following is one of the many possible answers to the question:

If $x \in R^+$ and $x < 2$, then

$$\lim_{n \text{ large}} \left\{ \frac{1}{\sqrt{x}} - \left[1 - \tfrac{1}{2}(x-1) + \frac{1 \times 3}{2 \times 4}(x-1)^2 - \cdots + (-1)^n \frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times (2n)}(x-1)^n \right] \right\} = 0.$$

Your answer may look very different from this, but it should include the following:

(i) the restriction that x must lie between 0 and 2;

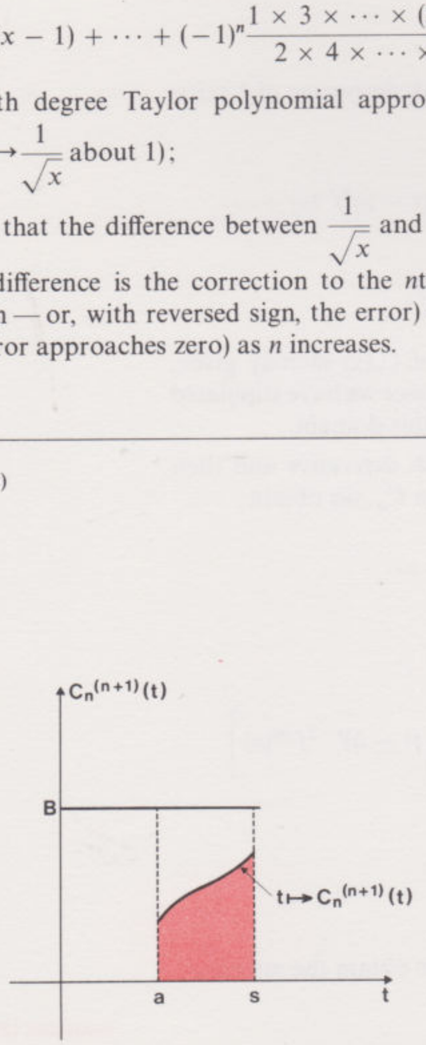
(ii) the polynomial

$$1 - \tfrac{1}{2}(x-1) + \cdots + (-1)^n \frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times (2n)}(x-1)^n$$

(this is the nth degree Taylor polynomial approximation to the function $x \longmapsto \dfrac{1}{\sqrt{x}}$ about 1);

(iii) the statement that the difference between $\dfrac{1}{\sqrt{x}}$ and the above polynomial (this difference is the correction to the nth degree Taylor approximation — or, with reversed sign, the error) approaches zero (that is, the error approaches zero) as n increases. ■

(continued from page 45)



Evaluating the integrals with the help of the Fundamental Theorem of Calculus (*Unit 13, Integration II*) gives, (since $C_n^{(n+1)} = DC_n^{(n)}$, by definition)

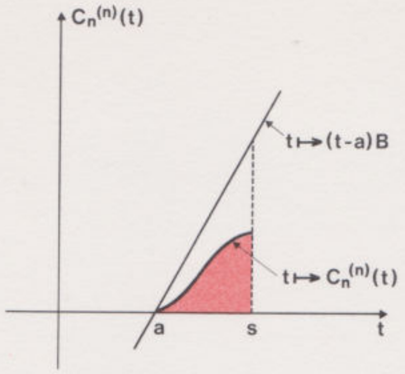
$$C_n^{(n)}(s) - C_n^{(n)}(a) \leqslant (s-a)B.$$

But the equation which we obtained earlier for $C_n^{(n)}(t)$ shows us that $C_n^{(n)}(a) = 0$, and since the last inequality holds for all s in $[a, x]$, it follows that:

$$C_n^{(n)}(t) \leqslant (t-a)B \qquad (t \in [a, x]).$$

Now we can repeat the procedure and reduce the order of the derivative of C_n one further. Integration from a to s, with $s \in [a, x]$, gives (see diagram):

$$\int_a^s C_n^{(n)} \leqslant \int_a^s (t \longmapsto (t-a)B)$$



That is,

$$C_n^{(n-1)}(s) - C_n^{(n-1)}(a) \leqslant \tfrac{1}{2}(s-a)^2 B.$$

But, once again, we have $C_n^{(n-1)}(a) = 0$, and since the last inequality holds for all s in $[a, x]$, it follows that:

$$C_n^{(n-1)}(t) \leqslant \tfrac{1}{2}(t-a)^2 B \qquad (t \in [a, x]).$$

Repeating the procedure $n - 1$ more times we obtain:

$$C_n^{(n-2)}(t) \leqslant \frac{1}{3!}(t-a)^3 B \qquad (t \in [a, x]),$$

$$C_n^{(n-3)}(t) \leqslant \frac{1}{4!}(t-a)^4 B \qquad (t \in [a, x]),$$

and finally

$$C_n(t) \leqslant \frac{1}{(n+1)!}(t-a)^{n+1} B \qquad (t \in [a, x]).$$

This is precisely the upper bound on $C_n(t)$ given by Taylor's Theorem in the case $x > a$ (since then we have $t \geqslant a$, so that $t - a$ is the same as $|t - a|$). Applying the same procedure for lower bounds, and for the case where $x < a$, we can complete the demonstration of the form of Taylor's Theorem given in the text.

The argument we have given is not, strictly speaking, a proof, since we have relied on diagrams to demonstrate results of the form

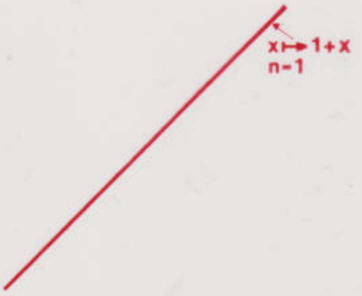
$$g(t) \leqslant h(t) \qquad (t \in [a, x])$$

implies

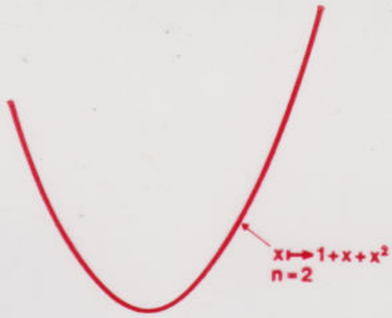
$$\int_a^x g \leqslant \int_a^x h.$$

It is not difficult to prove these results directly from the definition of an integral, but such proofs are really beyond the scope of the Foundation Course, since we have not put the properties of the real numbers on an axiomatic basis.

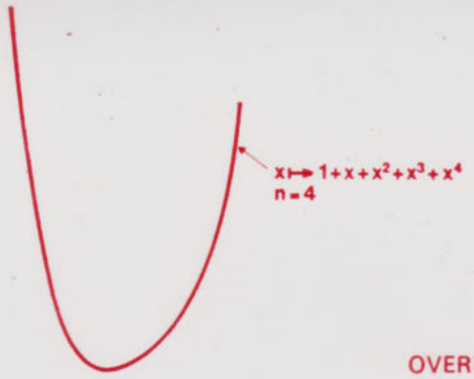
Unit No.		Title of Text
1		Functions
2		Errors and Accuracy
3		Operations and Morphisms
4		Finite Differences
5	NO TEXT	
6		Inequalities
7		Sequences and Limits I
8		Computing I
9		Integration I
10	NO TEXT	
11		Logic I — Boolean Algebra
12		Differentiation I
13		Integration II
14		Sequences and Limits II
15		Differentiation II
16		Probability and Statistics I
17		Logic II — Proof
18		Probability and Statistics II
19		Relations
20		Computing II
21		Probability and Statistics III
22		Linear Algebra I
23		Linear Algebra II
24		Differential Equations I
25	NO TEXT	
26		Linear Algebra III
27		Complex Numbers I
28		Linear Algebra IV
29		Complex Numbers II
30		Groups I
31		Differential Equations II
32	NO TEXT	
33		Groups II
34		Number Systems
35		Topology
36		Mathematical Structures



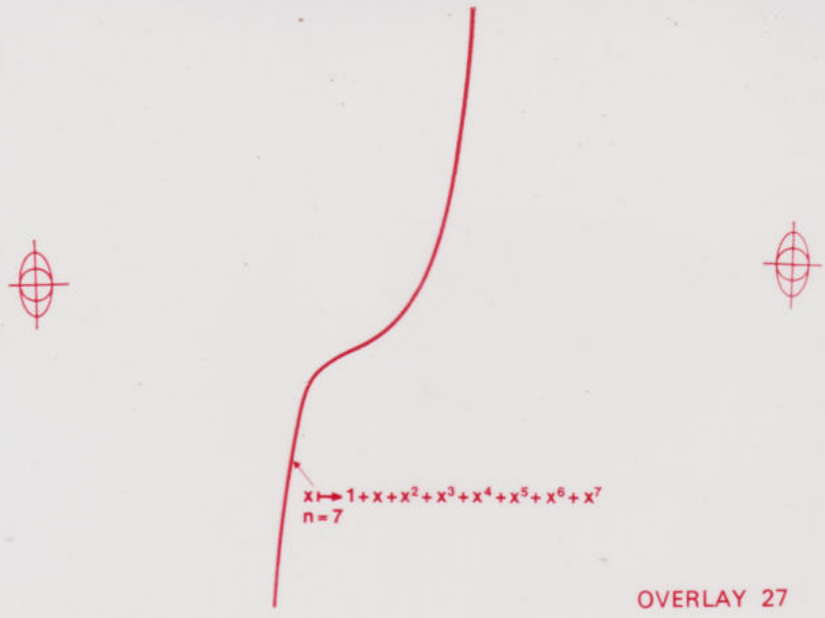
OVERLAY 24



OVERLAY 25

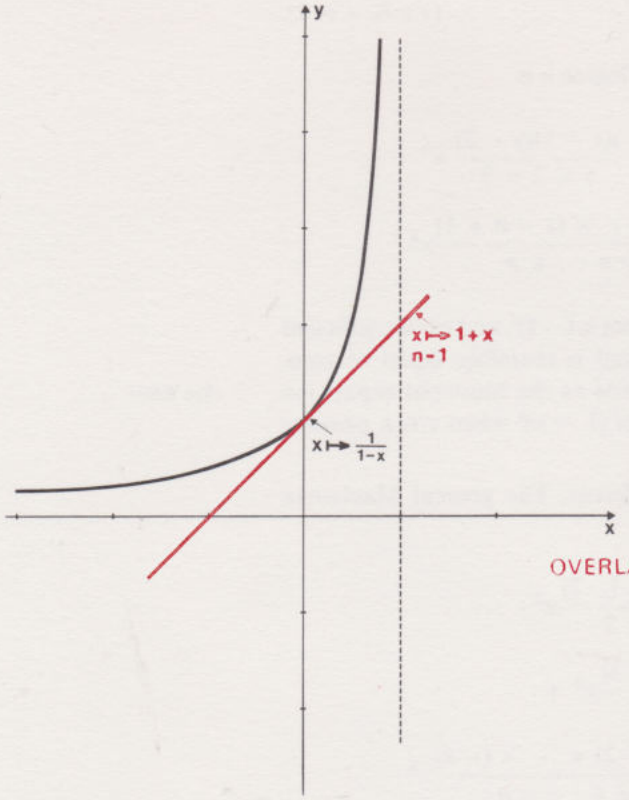


OVERLAY 26



OVERLAY 27

the successive Maclaurin approximations to $(1 - x)^{-1}$; these are shown in the following diagram* and its overlays:



OVERLAY 24

The graphs show that the nature of the approximation is not the same as in the case of $\sin x$: for $\sin x$ the interval over which the approximation is good gets wider as the degree of the polynomial gets higher; but for $(1 - x)^{-1}$ the interval of good approximation is always contained within the interval $[-1, 1]$.

These results show that the Taylor (Maclaurin) approximation method is quite temperamental: sometimes it is very effective, but on other occasions the approximations it produces are wide of the mark. The method is a very powerful one, but to be able to use it without getting into trouble one needs either very sound intuition or some theorems that will specify the situations in which the method is successful. In the next section of the text we shall leave the exploratory approach we have been using and look at the theory of the Taylor approximation method from a rigorous point of view.

* We have drawn only the part of the graph for which $1/(1 - x) > 0$; the dotted line is the line specified by $x = 1$.

The overlays are in the wallet on the inside of the back cover of this text.

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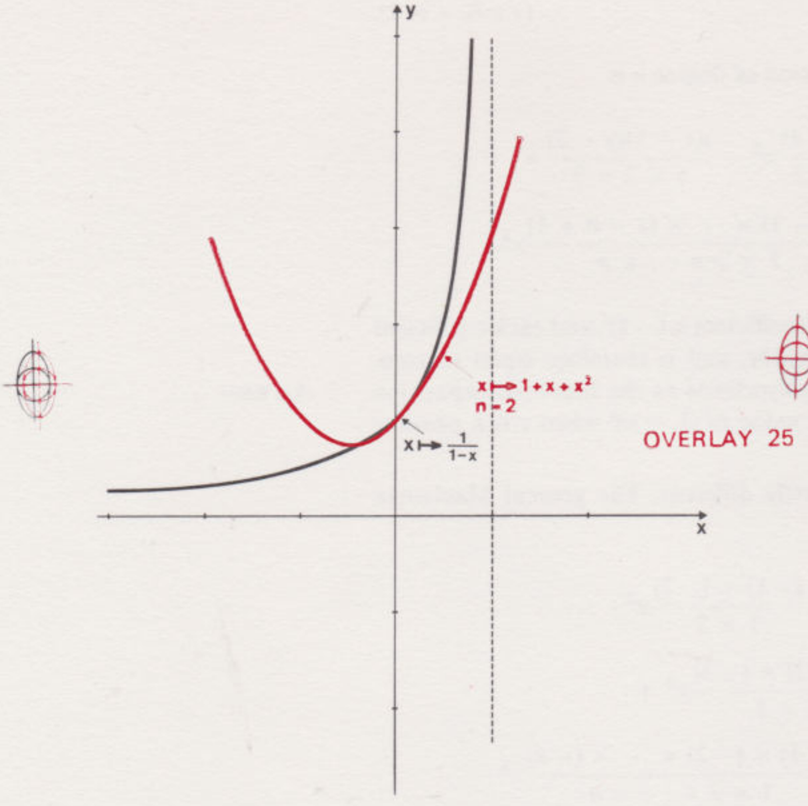

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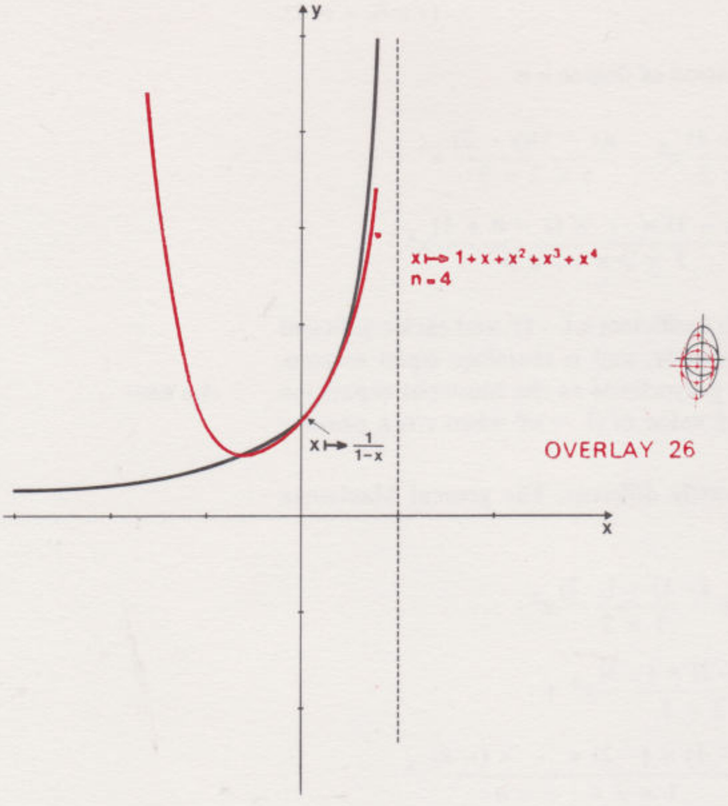

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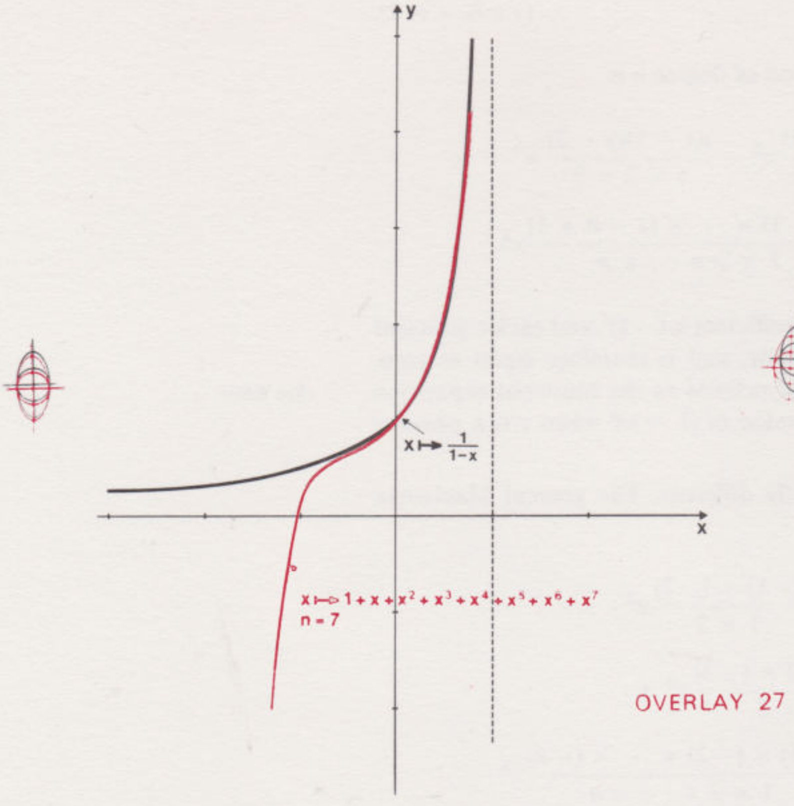

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